

LOCAL WELL-POSEDNESS AND LOW MACH NUMBER LIMIT OF THE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS IN CRITICAL SPACES

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ABSTRACT. The local well-posedness and low Mach number limit are considered for the multi-dimensional isentropic compressible viscous magnetohydrodynamic equations in critical spaces. First the local well-posedness of solution to the viscous magnetohydrodynamic equations with large initial data is established. Then the low Mach number limit is studied for general large data and it is proved that the solution of the compressible magnetohydrodynamic equations converges to that of the incompressible magnetohydrodynamic equations as the Mach number tends to zero. Moreover, the convergence rates are obtained.

1. INTRODUCTION

In this paper we consider the local well-posedness and low Mach number limit to the following isentropic compressible magnetohydrodynamic (MHD) equations in critical spaces (see [23, 24, 32]):

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (1.1)$$

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) \\ = H \cdot \nabla H - \frac{1}{2} \nabla(|H|^2) + \mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u, \end{aligned} \quad (1.2)$$

$$\partial_t H - \operatorname{curl}(u \times H) = -\operatorname{curl}(\nu \operatorname{curl} H), \quad \operatorname{div} H = 0, \quad (1.3)$$

$$(\rho, u, H)|_{t=0} = (\rho_0, u_0, H_0)(x), \quad x \in \mathbb{R}^d. \quad (1.4)$$

Here ρ denotes the density of the fluid, $u = (u^{(1)}, \dots, u^{(d)}) \in \mathbb{R}^d$ ($d = 2, 3$) is the fluid velocity field, $H = (H^{(1)}, \dots, H^{(d)}) \in \mathbb{R}^d$ is the magnetic field, and P is the pressure function satisfying $P'(\rho) > 0$. The constants $\mu > 0$ and λ denotes the shear and bulk viscosity coefficients of the flow, respectively, satisfying $2\mu + \lambda > 0$. The constant $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field.

The system (1.1)-(1.3) can be derived from the isentropic Navier-Stokes-Maxwell system by taking the zero dielectric constant limit [22]. Recently, many results on the system (1.1)-(1.3) were obtained. Li and Yu [25] obtained the optimal decay rate of smooth solution when the initial data is a small perturbation of some give constant state. Suen and Hoff [35] established the global weak solutions when the initial energy is small. Later,

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this result was extended to the case when the initial data may contain large oscillations or vacuum [26, 31]. Hu and Wang [19] obtained the global existence and large-time behavior of general weak solution with finite energy in the sense of [13, 14, 29]. Li, Su and Wang [27] obtained the local strong solution to (1.1)-(1.3) with large initial data. Suen [33] as well as Xu and Zhang [34] established some blow-up criteria for (1.1)-(1.3). The low Mach number limit of the system (1.1)-(1.3) has also been studied recently. Hu and Wang [18] proved the convergence of the weak solutions of the compressible MHD equations to a weak solution of the viscous incompressible MHD equations. Jiang, Ju and Li obtained the convergence of the weak solutions of the compressible MHD equations to the strong solution of the ideal incompressible MHD equations in the whole space [20] or to the viscous incompressible MHD equations in torus [21] for general initial data. Feireisl, Novotny, and Sun [15] extended and improved the results in [20] to the unbounded domain case. Li [28] studied the inviscid, incompressible limit of the viscous isentropic compressible MHD equations for local smooth solutions with well-prepared initial data. Dou, Jiang, and Ju [12] studied the low Mach number limit for the compressible magnetohydrodynamic equations in a bounded domain with perfectly conducting boundary. See the recent papers [12, 15, 20, 21, 28] and the references therein for more discussions of other related results.

We point out that all of the above results were carried out in the framework of Sobolev spaces. Obviously, up to a change of the pressure function P into $l^2 P$ in the system (1.1)-(1.3), it is invariant under the scaling:

$$\rho^\epsilon(t, x), u^\epsilon(t, x), H^\epsilon(t, x) \rightarrow \rho^\epsilon(l^2 t, lx), lu^\epsilon(l^2 t, lx), l^2 H^\epsilon(l^2 t, lx). \quad (1.5)$$

Thus it is natural to study the system (1.1)-(1.3) in critical spaces. (A function space $\mathcal{E} \in \mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$ is called a critical space for the system (1.1)-(1.3) if the associated norm is invariant under the transformation of (1.5) (up to a constant independent of l .) In [16], Hao obtained the global existence of solution to the system (1.1)-(1.3) in the critical space when the initial data is a small perturbation of some given constant state. In [30], the second author low Mach number limit of the system (1.1)-(1.3) for small initial data in Besov space. In [2], Bian and Yuan studied the inviscid version of (1.1)-(1.3) in the super critical Besov spaces.

The purpose of this paper is to study the local well-posedness and low Mach number limit of the system (1.1)-(1.3) with large initial data in critical Besov spaces in the whole space \mathbb{R}^d . We add the the following condition to the system (1.1)-(1.3) in the far field:

$$\rho \rightarrow 1, u \rightarrow 0, H \rightarrow 0 \text{ as } |x| \rightarrow \infty. \quad (1.6)$$

Denoting

$$a := \rho - 1,$$

introducing the viscosity operator

$$\mathcal{A} := \mu \Delta + (\lambda + \mu) \nabla \operatorname{div},$$

and using the identities:

$$\begin{aligned} \operatorname{curl} \operatorname{curl} H &= \nabla \operatorname{div} H - \Delta H, \\ \operatorname{curl}(u \times H) &= u(\operatorname{div} H) - H(\operatorname{div} u) + H \cdot \nabla u - u \cdot \nabla H, \end{aligned}$$

we can rewrite the Cauchy problem (1.1)-(1.4) as the following:

$$\partial_t a + u \cdot \nabla a = -(1 + a) \operatorname{div} u, \quad (1.7)$$

$$\partial_t u + u \cdot \nabla u - \frac{1}{1+a} \mathcal{A}u + \nabla G(a) = \frac{1}{1+a} \left(H \cdot \nabla H - \frac{1}{2} \nabla(|H|^2) \right), \quad (1.8)$$

$$\partial_t H + u \cdot \nabla H - H \cdot \nabla u - \nu \Delta H = -(\operatorname{div} u)H, \quad \operatorname{div} H = 0, \quad (1.9)$$

$$(a, u, H)|_{t=0} = (a_0, u_0, H_0)(x), \quad x \in \mathbb{R}^d, \quad (1.10)$$

where

$$\nabla G(a) := \frac{1}{1+a} \nabla P(1+a).$$

To state our results, we introduce the following function spaces:

$$\begin{aligned} E_T^\alpha &:= \tilde{C}_T(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}) \times (\tilde{C}_T(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha}))^d \\ &\quad \times (\tilde{C}_T(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha}))^d, \\ F_T^{\frac{d}{2}+\beta} &:= \mathcal{C}_b([0, T]; \dot{B}_{2,1}^{\frac{d}{2}+\beta-1})^{2d} \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta+1})^{2d} \end{aligned}$$

with

$$\tilde{C}_T(\dot{B}_{p,1}^s) := \mathcal{C}([0, T]; \dot{B}_{p,1}^s) \cap \tilde{L}_T^\infty(\dot{B}_{p,1}^s),$$

where $\dot{B}_{p,1}^s$ denotes the homogeneous Besov space. We shall explain these notations in detail in Appendix A.

Our first result of this paper reads as follows.

Theorem 1.1. *Assume that the initial data (a_0, u_0, H_0) satisfy $\operatorname{div} H_0 = 0$ and*

$$a_0 \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}, \quad u_0 \in \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}, \quad H_0 \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha},$$

for some $\alpha \in (0, 1]$. If, in addition, $\inf_{x \in \mathbb{R}^d} a_0(x) > -1$, then there exists a $T > 0$ such that the problem (1.7)-(1.10) has a unique solution (a, u, H) on $[0, T] \times \mathbb{R}^d$ which belongs to E_T^α and satisfies $\inf_{(t,x) \in [0,T] \times \mathbb{R}^d} a(t, x) > -1$.

Remark 1.1. Theorem 1.1 still holds for $\alpha = 0$. Here we assume additional regularity on the initial data to obtain more regular solution, which is needed in the study of the low Mach number limit to the system (1.1)-(1.3) below. For the case $\alpha = 0$, the proof of the uniqueness of solution in dimension two needs additional arguments, and we refer the readers to [8, 9] for the corresponding discussions on the isentropic Navier-Stokes equations.

Denote by ϵ the (scaled) Mach number. Introducing the scaling:

$$\rho(x, t) = \rho^\epsilon(x, \epsilon t), \quad u(x, t) = \epsilon u^\epsilon(x, \epsilon t), \quad H(x, t) = \epsilon H^\epsilon(x, \epsilon t),$$

and assuming that the viscosity coefficients μ , ξ , and ν are small constants and scaled as:

$$\mu = \epsilon \mu^\epsilon, \quad \lambda = \epsilon \lambda^\epsilon, \quad \nu = \epsilon \nu^\epsilon,$$

then we can rewrite the problem (1.1)-(1.4) as the following:

$$\partial_t \rho^\epsilon + \operatorname{div}(\rho^\epsilon u^\epsilon) = 0, \quad (1.11)$$

$$\begin{aligned} \partial_t(\rho^\epsilon u^\epsilon) + \operatorname{div}(\rho^\epsilon u^\epsilon \otimes u^\epsilon) + \frac{\nabla P^\epsilon}{\epsilon^2} \\ = H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla(|H^\epsilon|^2) + \mu^\epsilon \Delta u^\epsilon + (\mu^\epsilon + \lambda^\epsilon) \nabla \operatorname{div} u^\epsilon, \end{aligned} \quad (1.12)$$

$$\partial_t H^\epsilon + (\operatorname{div} u^\epsilon) H^\epsilon + u^\epsilon \cdot \nabla H^\epsilon - H^\epsilon \cdot \nabla u^\epsilon = \nu^\epsilon \Delta H^\epsilon, \quad \operatorname{div} H^\epsilon = 0, \quad (1.13)$$

$$(\rho^\epsilon, u^\epsilon, H^\epsilon)|_{t=0} = (\rho_0^\epsilon, u_0^\epsilon, H_0^\epsilon)(x), \quad x \in \mathbb{R}^d, \quad (1.14)$$

where $P^\epsilon := P(\rho^\epsilon)$ stands for the pressure. For the simplicity of notations and presentation, we shall assume that μ^ϵ , λ^ϵ , and ν^ϵ are constants, independent of ϵ , and still denote them as μ , λ , and ν with an abuse of notations.

Formally, if let ϵ go to zero, then we have $\nabla P^\epsilon \rightarrow 0$. Thus, if $P'(\cdot)$ does not vanish, the limit density has to be a constant. Denote by (v, B) the limit of (u^ϵ, H^ϵ) . Taking the limit in the mass equation (1.11) implies that the limit v is divergence-free. Passing to the limit in the equations (1.12) and (1.13), we conclude that (v, B) must satisfy the following incompressible MHD equations:

$$\partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \pi = B \cdot \nabla B - \frac{1}{2} \nabla(|B|^2), \quad (1.15)$$

$$\partial_t B + v \cdot \nabla B - B \cdot \nabla v = \nu \Delta B, \quad (1.16)$$

$$\operatorname{div} v = 0, \quad \operatorname{div} B = 0, \quad (1.17)$$

$$(v, B)|_{t=0} = (v_0, B_0)(x), \quad x \in \mathbb{R}^d. \quad (1.18)$$

As mentioned before, the rigorous derivation of the above heuristic process was proved recently in [18, 20, 21, 28] in the framework of Sobolev spaces. Here we want to justify the above formal process in critical Besov spaces. More precisely, we shall establish the convergence of the system (1.11)-(1.13) to the system (1.15)-(1.17) based on the results obtained in Theorem 1.1. We shall focus on the case of ill-prepared data where the acoustic waves caused by the oscillations must be considered.

Writing $\rho^\epsilon = 1 + \epsilon b^\epsilon$, it is easy to check that $(b^\epsilon, u^\epsilon, H^\epsilon)$ satisfies

$$\partial_t b^\epsilon + \frac{\operatorname{div} u^\epsilon}{\epsilon} = -\operatorname{div}(b^\epsilon u^\epsilon), \quad (1.19)$$

$$\begin{aligned} \partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon - \frac{\mathcal{A}u^\epsilon}{1 + \epsilon b^\epsilon} + \frac{P'(1 + \epsilon b^\epsilon) \nabla b^\epsilon}{(1 + \epsilon b^\epsilon) \epsilon} \\ = \frac{1}{1 + \epsilon b^\epsilon} (H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla(|H^\epsilon|^2)), \end{aligned} \quad (1.20)$$

$$\partial_t H^\epsilon + (\operatorname{div} u^\epsilon) H^\epsilon + u^\epsilon \cdot \nabla H^\epsilon - H^\epsilon \cdot \nabla u^\epsilon = \nu \Delta H^\epsilon, \quad \operatorname{div} H^\epsilon = 0, \quad (1.21)$$

$$(b^\epsilon, u^\epsilon, H^\epsilon)|_{t=0} = (b_0^\epsilon, u_0^\epsilon, H_0^\epsilon)(x), \quad x \in \mathbb{R}^d. \quad (1.22)$$

For the sake of simplicity, we shall also assume that the initial data $(b_0^\epsilon, u_0^\epsilon, H_0^\epsilon)$ does not depend on ϵ and will be denoted by (b_0, u_0, H_0) . The general case of $(b_0^\epsilon, u_0^\epsilon, H_0^\epsilon) \rightarrow (b_0, u_0, H_0)$ as $\epsilon \rightarrow 0$ in some Besov spaces can be treated similarly by a slight modification of the arguments presented here.

Denoting \mathcal{P} the Leray projector on solenoidal vector fields defined by $\mathcal{P} := I - \mathcal{P}^\perp$ with $\mathcal{P}^\perp := \Delta^{-1} \nabla \operatorname{div}$, and introducing the following functional space:

$$\begin{aligned} E_{\epsilon, T}^{\frac{d}{2}+\beta} &:= \mathcal{C}_b([0, T]; \tilde{B}_\epsilon^{\frac{d}{2}+\beta, \infty}) \cap L_T^1(\tilde{B}_\epsilon^{\frac{d}{2}+\beta, 1}) \\ &\quad \times (\mathcal{C}_b([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1+\beta}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta}))^d \\ &\quad \times (\mathcal{C}_b([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1+\beta}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta}))^d, \end{aligned}$$

with the norm

$$\begin{aligned} \|(\rho, u, H)\|_{E_{\epsilon, T}^{\frac{d}{2}+\beta}} &:= \|\rho\|_{L_T^\infty(\tilde{B}_\epsilon^{\frac{d}{2}+\beta, \infty}) \cap L_T^1(\tilde{B}_\epsilon^{\frac{d}{2}+\beta, 1})} + \|u\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} \\ &\quad + \|H\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})}, \end{aligned}$$

our second result of the paper can be stated as follows.

Theorem 1.2. *Let $T_0 \in (0, \infty]$. Assume that the initial data (b_0, u_0, H_0) satisfy $\operatorname{div} H_0 = 0$ and*

$$b_0 \in \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}, \quad (u_0, H_0) \in (\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})^{d+d}$$

with $\alpha \in (0, 1/2)$ if $d = 3$ or $\alpha \in (0, 1/6]$ if $d = 2$. Suppose that the incompressible system (1.17)-(1.18) with initial data $(\mathcal{P}u_0, H_0)$ has a solution $(v, B) \in F_{T_0}^{\frac{d}{2}} \cap F_{T_0}^{\frac{d}{2}+\alpha}$. Let $V := \|(v, B)\|_{F_{T_0}^{\frac{d}{2}} \cap F_{T_0}^{\frac{d}{2}}}$ and

$$X_0 := \|b_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}} + \|\mathcal{P}^\perp u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}} + \|H_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}},$$

then there exist two positive constants ϵ_0 and C , depending only on $d, \alpha, \lambda, \mu, \nu, P, V$, and X_0 , such that the following results hold true:

- (i) *For all $\epsilon \in (0, \epsilon_0]$, the problem (1.19)-(1.22) has a unique global solution $(b^\epsilon, u^\epsilon, H^\epsilon)$ in $E_{\epsilon, T_0}^{\frac{d}{2}} \cap E_{\epsilon, T_0}^{\frac{d}{2}+\alpha}$ such that*

$$\|(b^\epsilon, u^\epsilon, H^\epsilon)\|_{E_{\epsilon, T_0}^{\frac{d}{2}} \cap E_{\epsilon, T_0}^{\frac{d}{2}+\alpha}} \leq C;$$

- (ii) *$(\mathcal{P}u^\epsilon, H^\epsilon) \rightarrow (v, B)$ in $F_{T_0}^{\frac{d}{2}} \cap F_{T_0}^{\frac{d}{2}+\alpha}$ as $\epsilon \rightarrow 0$, and*

$$\|\mathcal{P}u^\epsilon - v\|_{F_{T_0}^{\frac{d}{2}} \cap F_{T_0}^{\frac{d}{2}+\alpha}} \leq C\epsilon^{\frac{2\alpha}{2+d+2\alpha}},$$

$$\|H^\epsilon - B\|_{F_{T_0}^{\frac{d}{2}} \cap F_{T_0}^{\frac{d}{2}+\alpha}} \leq C\epsilon^{\frac{2\alpha}{2+d+2\alpha}};$$

- (iii) *$(b^\epsilon, \mathcal{P}^\perp u^\epsilon)$ tends to $(0, 0)$ as $\epsilon \rightarrow 0$ in the following sense:*

$$\|(b^\epsilon, \mathcal{P}^\perp u^\epsilon)\|_{L_T^p(\dot{B}_{\infty,1}^{\alpha-1+\frac{1}{p}})} \leq C\epsilon^{\frac{1}{p}} \quad \text{if } d = 3 \text{ and } 2 < p < \infty,$$

$$\|(b^\epsilon, \mathcal{P}^\perp u^\epsilon)\|_{L_T^4(\dot{B}_{\infty,1}^{\alpha-\frac{3}{4}})} \leq C\epsilon^{\frac{1}{4}} \quad \text{if } d = 2.$$

Remark 1.2. The regularity assumption on the solution of the incompressible system (1.17)-(1.18) is reasonable. Since we can not find it in the literature, we shall present a brief proof in Proposition B.1 of the Appendix B.

Remark 1.3. In Theorem 1.2, we need the additional constraint on the index α since it provides some decay in ϵ , which is used in many places of the proof.

Remark 1.4. For the case $d = 2$, one can choose $T_0 = +\infty$ in Theorem 1.2.

Remark 1.5. Due to the absence of disperse effects on the oscillation equations, it is more complicated and difficult to study the low Mach number limit of the system (1.11)-(1.13) for the period case in Besov spaces. We shall report this result in a forthcoming paper.

We now recall a few closely related results on the isentropic Navier-Stokes equations (i.e., $H = 0$ in the system (1.1)-(1.3)). In Danchin [6] the global well-posedness of isentropic Navier-Stokes equations in the critical Besov space was first obtained when the initial data is a small perturbation around some given constant state, and recently, the results of [6] were extended to more general Besov spaces in [3, 5, 17]. In a series of papers by Danchin [7–9], the local well-posedness of solutions to the isentropic Navier-Stokes equations with large initial data was proved. In [10, 11], the zero Mach number limit of the isentropic Navier-Stokes equations in the whole space or torus with ill-prepared initial data was studied.

Next we give some comments on the proofs of Theorems 1.1 and 1.2. We remark that when $H = 0$ our results coincide with the results obtained by Danchin [8–10] on the isentropic Navier-Stokes equations, hence extend some results in [8–10] to the isentropic compressible MHD equations. In our proofs of Theorems 1.1 and 1.2, we use some ideas developed in [8–10]. Besides the difficulties mentioned in [8–10], here the main difficulty is the strong coupling of the velocity and the magnetic field. We shall deal with them in detail in the proofs of Theorems 1.1 and 1.2. More precisely, in the proof of Theorem 1.1, we introduce a linearized version of the equation for the magnetic field (2.3) and obtain a tame estimate of the solution, and the coupling terms of the velocity and the magnetic field are analyzed in detail in each step of the proof of Theorem 1.1; see especially the proof of Proposition 2.5 in Section 2 below. In the proof of Theorem 1.2, several new systems analogous to the incompressible MHD equations (see the systems (B.3) and (B.6) and Propositions B.2 and B.3 below) are introduced and studied to establish the estimates on the incompressible part of the original compressible MHD equations, and also the coupling terms of the velocity and the magnetic field are analyzed in detail in each step of the proof of Theorem 1.2; see Section 3 below. In particular, the special structure of the isentropic compressible MHD system are fully utilized in our analysis.

Our paper is organized as follows. In Section 2, we establish the local existence and uniqueness of solution to the problem (1.7)-(1.10). In Section 3, we discuss the low Mach limit of the problem (1.19)-(1.22). We close our paper with two appendices. In Appendix A, we define some functional spaces (homogeneous and hybrid Besov spaces), recall some basic tools on paradifferential calculus and state some tame estimates for composition or product. Finally, in Appendix B, we present the regularity results on incompressible MHD equations (1.15)-(1.17) and the analogies which are needed in the proof of Theorem 1.2.

2. LOCAL WELL-POSEDNESS OF THE COMPRESSIBLE MHD EQUATIONS

In this section we shall establish the local well-posedness of the compressible MHD equations (1.7)-(1.9). We shall follow and adapt the methods developed by Danchin in [7–9] (see also [1]). We shall focus on the analysis of the coupling terms of the velocity field u and the magnetic field H . We divide this section into four parts. First, we recall some basic results on the linear transport equation and prove a result on the linearized magnetic field equation and a result on the smooth solution of the system (1.7)-(1.9) which is new in some sense and play an essential role in our proof. Next, we establish the local existence of the solution. Third, we discuss the uniqueness of the solution. Finally, we state a continuation criterion of the solution.

Letting I be an interval of \mathbb{R} and X be a Banach space, we use the notation $\mathcal{C}_b(I, X)$ to denote the set of bounded and continuous function on I with values in X . Similarly, $L^r(I, X)$ is used to denote the set of measurable functions on I valued in X such that the map $t \rightarrow \|u(t)\|_X$ belongs to the Lebesgue space $L^r(I)$. If $I = [0, T]$, we shall abbreviate $L^r(I, X)$ as $L_T^r(X)$. For any $p \geq 1$ we use p' to denote the conjugate exponent of p , defined by $\frac{1}{p} + \frac{1}{p'} = 1$. We use the letter C to denote the positive constant which may change from line to line. We also omit the spatial domain \mathbb{R}^d ($d = 2, 3$) in the integrals and the norms of function spaces for simplicity of presentation.

2.1. A priori estimates for the linearized equations. Let us first recall the standard estimates in the Besov spaces for the following linear transport equation:

$$\partial_t a + v \cdot \nabla a = f, \quad a|_{t=0} = a_0. \quad (2.1)$$

Proposition 2.1 ([1]). *Let $\sigma \in (-\frac{d}{2}, \frac{d}{2}]$. There exists a constant C , depending only on d and σ , such that for all solution $a \in L_T^\infty(\dot{B}_{2,1}^\sigma)$ of (2.1), initial data a_0 in $\dot{B}_{2,1}^\sigma$, and f in $L_T^1(\dot{B}_{2,1}^\sigma)$, we have, for a.e. $t \in [0, T]$,*

$$\|a\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^\sigma)} \leq \left\{ \|a_0\|_{\dot{B}_{2,1}^\sigma} + \int_0^t e^{-CV(\tau)} \|f(\tau)\|_{\dot{B}_{2,1}^\sigma} d\tau \right\} e^{CV(t)}$$

with

$$V := \int_0^t V'(\tau) d\tau$$

and

$$V'(t) := \|\nabla v(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Proposition 2.2 ([1]). *Let $1 \leq p \leq p_1 \leq \infty$. Assume that*

$$\sigma > -d \min \left\{ \frac{1}{p_1}, \frac{1}{p'} \right\} \quad \text{or} \quad \sigma > -1 - d \min \left\{ \frac{1}{p_1}, \frac{1}{p'} \right\} \quad \text{if } \operatorname{div} v = 0$$

with the additional condition $\sigma \leq 1 + \frac{d}{p_1}$. Let $a_0 \in \dot{B}_{p,r}^\sigma$, $f \in L_T^1(\dot{B}_{p,1}^\sigma)$ and v be a time-dependent vector field such that $v \in L_T^\rho(\dot{B}_{\infty,\infty}^{-M})$ for some $\rho > 1$ and $M > 0$ and

$$\nabla v \in L_T^1(\dot{B}_{p_1,\infty}^{p_1} \cap L^\infty) \quad \text{if } \sigma < 1 + \frac{d}{p_1}.$$

Then the problem (2.1) has a unique solution a in the space $\mathcal{C}([0, T]; \dot{B}_{p,1}^\sigma)$.

For the momentum equation (1.8), we have to consider a linearization which allows for non-constant coefficients, namely,

$$\partial_t u + v \cdot \nabla u + u \cdot \nabla w - b \mathcal{A} u = g, \quad u|_{t=0} = u_0, \quad (2.2)$$

where b is a given positive function depending on (t, x) and tending to some constant (say 1) when x goes to infinity.

Proposition 2.3 ([1]). *Let $\alpha \in (0, 1]$ and $s \in (-\frac{d}{2}, \frac{d}{2}]$. Assume that $b = 1 + c$ with $c \in L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2} + \alpha})$ and that*

$$b_* := \inf_{(t,x) \in [0,T] \times \mathbb{R}^d} b(t, x) > 0.$$

Assume that $u_0 \in \dot{B}_{2,1}^s$, $g \in L_T^1(\dot{B}_{2,1}^s)$, and $v, w \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})$ are time-dependent vector fields. Then there exists a universal constant κ , and a constant C depending only on d, α , and s , such that, for all $t \in [0, T]$, the solution of the problem (2.2) satisfies

$$\begin{aligned} & \|u\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^s)} + \kappa b_* \mu \|u\|_{L_t^1(\dot{B}_{2,1}^{s+2})} \leq (\|u_0\|_{\dot{B}_{2,1}^s} + \|g\|_{L_t^1(\dot{B}_{2,1}^s)}) \\ & \times \exp \left\{ C \int_0^t \left(\|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|w\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + b_* \mu \left(\frac{2\mu + \lambda}{b_* \mu} \right)^{\frac{2}{\alpha}} \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}+\alpha}} \right) d\tau \right\}. \end{aligned}$$

For the linearized magnetic equation associated with the system (1.7)-(1.10):

$$\begin{cases} \partial_t H + u \cdot \nabla H + H \cdot \nabla u - \nu \Delta H = -(\operatorname{div} u)H + g, \\ \operatorname{div} H = 0, \\ H|_{t=0} = H_0, \end{cases} \quad (2.3)$$

we have the following proposition.

Proposition 2.4. *Let $s \in (-\frac{d}{2}, \frac{d}{2}]$. Assume that $H_0 \in \dot{B}_{2,1}^s$ with $\operatorname{div} H_0 = 0$, $g \in L_T^1(\dot{B}_{2,1}^s)$, and $u \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})$ is time-dependent vector field. Then there exist a universal constant κ , and a constant C depending only on d and s , such that*

$$\|H\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^s)} + \kappa \nu \|H\|_{L_t^1(\dot{B}_{2,1}^{s+2})} \leq (\|H_0\|_{\dot{B}_{2,1}^s} + \|g\|_{L^1(\dot{B}_{2,1}^s)}) \exp \left\{ C \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau \right\}.$$

Proof. We shall adopt the homogeneous Littlewood-Paley decomposition technique to obtain the desired estimate. More precisely, by applying $\dot{\Delta}_j$ to (2.3), we obtain that

$$\partial_t H_j + u \cdot \nabla H_j - \nu \Delta H_j = -\dot{\Delta}_j((\operatorname{div} u)H) - \dot{\Delta}_j(H \cdot \nabla u) + R_j + g_j, \quad H_j|_{t=0} = H_{0j},$$

where

$$H_j := \dot{\Delta}_j H, \quad R_j := \sum_k [u^k, \dot{\Delta}_j] \partial_k H, \quad g_j := \dot{\Delta}_j g, \quad H_{0j} := \dot{\Delta}_j H_0.$$

Taking the L^2 inner product of the above equation with H_j , we easily get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|H_j\|_{L^2}^2 - \frac{1}{2} \int |H_j|^2 \operatorname{div} u dx + \nu \int |\nabla H_j|^2 dx \\ & \leq \|H_j\|_{L^2} (\|\dot{\Delta}_j(\operatorname{div} u H)\|_{L^2} + \|\dot{\Delta}_j(H \cdot \nabla u)\|_{L^2} + \|R_j\|_{L^2} + \|g_j\|_{L^2}). \end{aligned}$$

Hence, by the Bernstein's inequality, we get, for some universal constant κ ,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|H_j\|_{L^2}^2 + 2\kappa \nu 2^{2j} \|\nabla H_j\|_{L^2} \\ & \leq \|H_j\|_{L^2} (\|\dot{\Delta}_j(\operatorname{div} u H)\|_{L^2} + \|\dot{\Delta}_j(H \cdot \nabla u)\|_{L^2} \\ & \quad + \|R_j\|_{L^2} + \frac{1}{2} \|\operatorname{div} u\|_{L^\infty} \|H_j\|_{L^2} + \|g_j\|_{L^2}). \end{aligned} \quad (2.4)$$

Thanks to Proposition A.2 and the commutator estimates (see Lemma 2.100 in [1]), we have the following estimates for $\dot{\Delta}_j(\operatorname{div} u H)$, $\dot{\Delta}_j(H \cdot \nabla u)$, and R_j :

$$\|\dot{\Delta}_j(\operatorname{div} u H)\|_{L^2} \leq C c_j 2^{-js} \|\operatorname{div} u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|H\|_{\dot{B}_{2,1}^s}, \quad \text{if } -\frac{d}{2} < s \leq \frac{d}{2},$$

$$\begin{aligned}\|\dot{\Delta}_j(H \cdot \nabla u)\|_{L^2} &\leq C c_j 2^{-js} \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|H\|_{\dot{B}_{2,1}^s}, \quad \text{if } -\frac{d}{2} < s \leq \frac{d}{2}, \\ \|R_j\|_{L^2} &\leq C c_j 2^{-js} \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|H\|_{\dot{B}_{2,1}^s}, \quad \text{if } -\frac{d}{2} < s \leq \frac{d}{2} + 1,\end{aligned}$$

where $(c_j)_{j \in \mathbb{Z}}$ denotes a positive sequence such that $\sum_{j \in \mathbb{Z}} c_j = 1$.

Formally dividing both sides of the inequality (2.4) by $\|H_j\|_{L^2}$ and integrating over $[0, t]$, one has

$$\begin{aligned}\|H_j(t)\|_{L^2} + 2\kappa\nu 2^{2j} \int_0^t \|H_j\|_{L^2} d\tau \\ \leq \|H_j(0)\|_{L^2} + \|g_j\|_{L^2} + C 2^{-js} \int_0^t c_j \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|H\|_{\dot{B}_{2,1}^s} d\tau.\end{aligned}$$

Now, multiplying both sides by 2^{js} and summing over j , we end up with

$$\|H\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^s)} + \kappa\nu \|H\|_{L_t^1(\dot{B}_{2,1}^{s+2})} \leq \|H(0)\|_{\dot{B}_{2,1}^s} + \|g_j\|_{L^2} + C \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|H\|_{\dot{B}_{2,1}^s} d\tau$$

for some constant C depending only on d and s . Applying Gronwall's lemma then completes the proof. \square

With these estimates in hand, we can prove the following result for smooth solutions to the problem (1.7)-(1.10).

Proposition 2.5. *Let (a, u, H) satisfy (1.7)-(1.10) on $[0, T] \times \mathbb{R}^d$. Suppose that there exist two positive constants, b_* and b^* , such that*

$$b_* \leq 1 + a_0 \leq b^*$$

and that $a \in \mathcal{C}^1([0, T]; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})$ and $(u, H) \in \mathcal{C}^1([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})^{d+d}$. Assume, in addition, that there exists a function $u_L \in \mathcal{C}^1([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})^d$ satisfies

$$\partial_t u_L - \mathcal{A}u_L = 0, \quad u_L|_{t=0} = u_0,$$

and that there exists a function $H_L \in \mathcal{C}^1([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})^d$ satisfies

$$\partial_t H_L - \nu \Delta H_L = 0, \quad H_L|_{t=0} = H_0.$$

Denote $\bar{u} := u - u_L$ and

$$\begin{aligned}A_0^\alpha &:= \|a_0\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}}, \quad A^\alpha(t) := \|a\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})}, \\ U_0^\alpha(t) &:= \|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}}, \quad U_L^\alpha(t) := \|u_L\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})}, \\ \bar{U}^\alpha(t) &:= \|\bar{u}\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} + b_* \mu \|\bar{u}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})}, \\ H_0^\alpha &:= \|H_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}}, \quad H_L(t) := \|H_L\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}, \\ H^\alpha(t) &:= \|H\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} + \nu \|H\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})}.\end{aligned}$$

Assume further that there exist two constant η and C , depending only on d, α , and G , such that

$$b_* \mu \left(\frac{\bar{\nu}}{b_* \mu} \right)^{\frac{2}{\alpha}} T (A_0^\alpha + 1)^{\frac{2}{\alpha}} \leq \eta, \quad (2.5)$$

$$(1 + A_0^\alpha)^2 \left(T(1 + (H_0^\alpha)^2) + \bar{\nu} U_L^\alpha(T) + (U_0^\alpha + \bar{\nu})(H_0^\alpha)^2 U_L^\alpha(T) \right. \\ \left. + (U_0^\alpha + (H_0^\alpha)^2 + 1)(H_0^\alpha)^{\frac{3}{2}} H_L^{\frac{1}{2}}(T) \right) \leq \eta b_* \mu, \quad (2.6)$$

then we have

$$\frac{b_*}{2} \leq 1 + a(t, x) \leq 2b^* \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^d, \quad (2.7)$$

$$A^\alpha(T) \leq 2A_0^\alpha + 1, \quad H^\alpha(T) \leq 2H_0^\alpha, \quad (2.8)$$

$$\bar{U}^\alpha(T) \leq C(1 + A_0^\alpha)^2 \left(T(1 + (H_0^\alpha)^2) + \bar{\nu} U_L^\alpha(T) + (U_0^\alpha + \bar{\nu})(H_0^\alpha)^2 U_L^\alpha(T) \right. \\ \left. + (U_0^\alpha + (H_0^\alpha)^2 + 1)(H_0^\alpha)^{\frac{3}{2}} H_L^{\frac{1}{2}}(T) \right), \quad (2.9)$$

with $\bar{\nu} := \lambda + 2\mu$.

Proof. Setting $\bar{H} := H - H_L$ and $I(a) := \frac{a}{1+a}$, we may write the system satisfied by (a, \bar{u}, H, \bar{H}) as:

$$\begin{cases} \partial_t a + u \cdot \nabla a + (1+a) \operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + \bar{u} \cdot \nabla u_L - \frac{1}{1+a} \mathcal{A} \bar{u} \\ \quad = -u_L \cdot \nabla u_L - I(a) \mathcal{A} u_L - \nabla G(a) + \frac{1}{1+a} (H \cdot \nabla H - \frac{1}{2} \nabla |H|^2), \\ \partial_t H + u \cdot \nabla H - H \cdot \nabla u - \nu \Delta H = -\operatorname{div} u H, \quad \operatorname{div} H = 0, \\ \partial_t \bar{H} + u \cdot \nabla \bar{H} - \bar{H} \cdot \nabla u - \nu \Delta \bar{H} \\ \quad = -\operatorname{div} u \bar{H} - \operatorname{div} u H_L - u \cdot \nabla H_L - H_L \cdot \nabla u, \\ (a, \bar{u}, H, \bar{H})|_{t=0} = (a_0, 0, H_0, 0), \end{cases} \quad (2.10)$$

We first estimate the bound of a . Applying the product law in Besov spaces, we get

$$\|(1+a) \operatorname{div} u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}} \leq C \left(1 + \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}} \right) \|\operatorname{div} u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}}.$$

Hence, combining Proposition 2.1 with Gronwall's lemma yields, for all $t \in [0, T]$,

$$A^\alpha(t) \leq A_0^\alpha \exp \left\{ C \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}} d\tau \right\} \\ + \exp \left\{ C \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}} d\tau \right\} - 1. \quad (2.11)$$

In order to ensure that the condition (2.7) is satisfied, we use the fact:

$$(\partial_t + u \cdot \nabla)(1+a)^{\pm 1} \pm (1+a)^{\pm 1} \operatorname{div} u = 0.$$

Hence, taking advantage of Gronwall's lemma, we obtain that

$$\|(1+a)^{\pm 1}(t)\|_{L^\infty} \leq \|(1+a_0)^{\pm 1}\|_{L^\infty} \exp \left\{ \int_0^t \|\operatorname{div} u\|_{L^\infty} d\tau \right\}.$$

Therefore, the condition (2.7) is satisfied on $[0, t]$ if

$$\int_0^t \|\operatorname{div} u\|_{L^\infty} d\tau \leq \log 2. \quad (2.12)$$

Now, we estimate H and \bar{H} . By Proposition 2.4 and Remark A.1, we have

$$\begin{aligned} H^\alpha(t) &\leq CH_0^\alpha(t) \exp \left\{ C \left(\int_0^t \|u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} d\tau + \int_0^t \|\bar{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} d\tau \right) \right\}, \\ \|H_L\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|H_L\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} &\leq CH_0^\alpha, \\ \|u_L\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|u_L\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} &\leq CU_0^\alpha. \end{aligned}$$

Thanks to Proposition A.2, we obtain that

$$\begin{aligned} \|u \cdot \nabla H_L\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} &\leq C \|u\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \|H_L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ &\leq C(U_0^\alpha + \bar{U}^\alpha(T))(H_0^\alpha)^{\frac{1}{2}} \|H_L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^{\frac{1}{2}}, \\ \|H_L \cdot \nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} &\leq C \|u\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|H_L\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \\ &\leq C \|u\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^{\frac{1}{2}} \|u\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^{\frac{1}{2}} \|H_L\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})}^{\frac{1}{2}} \|H_L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^{\frac{1}{2}} \\ &\leq C(U_0^\alpha + \bar{U}^\alpha(T))(H_0^\alpha)^{\frac{1}{2}} \|H_L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\bar{H}\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} &\leq C \left(\|H_L \cdot \nabla u\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|u \cdot \nabla H_L\|_{L_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \right) \exp \left\{ C \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} d\tau \right\} \\ &\leq C(U_0^\alpha + \bar{U}^\alpha(T))(H_0^\alpha)^{\frac{1}{2}} \|H_L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^{\frac{1}{2}} \exp \left\{ C \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} d\tau \right\}. \end{aligned}$$

In order to bound \bar{u} , we use Proposition 2.3 with $c = -I(a)$. Thanks to Proposition A.2, we have, for all $\beta \in \{0, \alpha\}$,

$$\begin{aligned} \|u_L \cdot \nabla u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} &\leq C \|\nabla u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}}, \\ \|\nabla G(a)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} &\leq C \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}+\beta}}, \\ \|I(a) \cdot \mathcal{A}u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} &\leq C \bar{\nu} \|I(a)\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\beta}} \|\nabla^2 u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}}, \\ \|I(a)\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\beta}} &\leq C \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\beta}}, \\ \left\| \frac{1}{1+a} (H \cdot \nabla H - \frac{1}{2} \nabla |H|^2) \right\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} &\leq C (1 + \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\nabla H\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|H\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}}. \end{aligned}$$

It is easily prove that

$$\|u_L\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \leq CU_0^\alpha.$$

Thus, we have

$$\begin{aligned} \bar{U}^\alpha(T) \leq & C \exp \left\{ C \int_0^T \left(\|\bar{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|u_L\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + b_* \mu \left(\frac{\bar{\nu}}{b_* \mu} \right)^\frac{2}{\alpha} \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}+\alpha}}^\frac{2}{\alpha} \right) dt \right\} \\ & \times \left(\|u_L\|_{L^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \|u_L\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \right. \\ & + \|a\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})} \left(T + \bar{\nu} \|u_L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})} \right) \\ & \left. + \left(1 + \|a\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})} \right) \|H\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \|H\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \right). \quad (2.13) \end{aligned}$$

Now, if T is sufficiently small so that

$$\exp\{CU_L^\alpha(T)\} \leq \sqrt{2}, \quad \exp\left\{C \frac{\bar{U}^\alpha(T)}{b_* \mu}\right\} \leq \sqrt{2}, \quad (2.14)$$

$$\exp\left\{Cb_* \mu \left(\frac{\bar{\nu}}{b_* \mu}\right)^\frac{2}{\alpha} T (A^\alpha(T))^\frac{2}{\alpha}\right\} \leq 2, \quad (2.15)$$

we have:

$$A^\alpha(T) \leq 2A_0^\alpha + 1, \quad (2.16)$$

$$\bar{U}^\alpha(T) \leq C(U_0^\alpha U_L^\alpha(T) + (1 + A_0^\alpha)(T + \bar{\nu} U_L^\alpha(T) + (H_0^\alpha)^2)) := CX(T), \quad (2.17)$$

$$\|\bar{H}\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \leq C(U_0^\alpha + X(T))(H_0^\alpha)^\frac{1}{2} \|H_L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\frac{1}{2}, \quad (2.18)$$

$$\|H\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \leq C(U_0^\alpha + X(T) + 1)(H_0^\alpha)^\frac{1}{2} \|H_L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^\frac{1}{2}. \quad (2.19)$$

By (2.13), (2.14), (2.15), and (2.19), we obtain that

$$\begin{aligned} \bar{U}^\alpha(T) & \leq C(U_0^\alpha U_L^\alpha(T) + (1 + A_0^\alpha)(T + \bar{\nu} U_L^\alpha(T)) \\ & \quad + (1 + A_0^\alpha)(U_0^\alpha + X(T) + 1)(H_0^\alpha)^\frac{3}{2} H_L(T)^\frac{1}{2}) \\ & \leq C(1 + A_0^\alpha)^2 (T(1 + (H_0^\alpha)^2) + \bar{\nu} U_L^\alpha(T) + (U_0^\alpha + \bar{\nu})(H_0^\alpha)^2 U_L^\alpha(T) \\ & \quad + (U_0^\alpha + (H_0^\alpha)^2 + 1)(H_0^\alpha)^\frac{3}{2} H_L^\frac{1}{2}(T)). \end{aligned}$$

Thus, if we choose $T > 0$ such that (2.5)-(2.6) is satisfied for some sufficiently small constant η , then both (2.9) and the above conditions (2.14)-(2.15) are satisfied with a strict inequality. It is now easy to complete the proof by means of a bootstrap argument. \square

2.2. Existence of the local solution. In this subsection, we shall prove the existence part of Theorem 1.1. We adopt the similar process developed by Danchin [8,9] for the compressible Navier-Stokes equations (see also [1]). Briefly, this process can be described as follows. First, we approximate the system (1.7)-(1.9) by a sequence of ordinary differential equations by applying the Friedrichs regularity method. Then, we prove uniform a priori estimates in E_T^α for these solutions. Next, we establish further boundedness properties involving the Hölder regularity with respect to time for these approximate solutions. Finally, we use the previous steps to show compactness and convergence of the approximate

solutions (up to an extraction). We shall focus on the analysis on the coupling term of the velocity field and the magnetic field.

2.2.1. Friedrichs approximation of the system. Let \dot{L}_n^2 be the set of L^2 functions spectrally supported in the annulus $\mathcal{C}_n := \{\xi \in \mathbb{R}^d \mid n^{-1} \leq \xi \leq n\}$ and let Ω_n be the set of functions (a, u, H) of $(\dot{L}_n^2)^{2d+1}$ such that $\inf_{x \in \mathbb{R}^d} a > -1$. The linear space \dot{L}_n^2 is endowed with the standard L^2 topology. Due to the Bernstein's inequality, the L^∞ topology on \dot{L}_n^2 is weaker than the usual L^2 topology, thus Ω_n is an open set of $(\dot{L}_n^2)^{2d+1}$. Let

$$\dot{\mathbb{E}}_n : L^2 \rightarrow \dot{L}_n^2$$

be the Friedrichs projector, defined by

$$\mathcal{F}\dot{\mathbb{E}}_n U(\xi) := \mathbf{1}_{\mathcal{C}_n}(\xi) \mathcal{F}U(\xi) \quad \text{for all } \xi \in \mathbb{R}^d.$$

We aim to solve the system of ordinary differential equations:

$$\frac{d}{dt} \begin{pmatrix} a \\ \bar{u} \\ H \end{pmatrix} = \begin{pmatrix} F_n(a, \bar{u}, H) \\ G_n(a, \bar{u}, H) \\ Q_n(a, \bar{u}, H) \end{pmatrix}, \quad \begin{pmatrix} a \\ \bar{u} \\ H \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} \dot{\mathbb{E}}_n a_0 \\ 0 \\ \dot{\mathbb{E}}_n H_0 \end{pmatrix} \quad (2.20)$$

with $u := \bar{u} + u_L$ and

$$\begin{aligned} F_n(a, \bar{u}, H) &:= -\dot{\mathbb{E}}_n \operatorname{div}((1+a)u), \\ G_n(a, \bar{u}, H) &:= \dot{\mathbb{E}}_n \left(\frac{1}{1+a} \mathcal{A} \bar{u} \right) - \dot{\mathbb{E}}_n(u \cdot \nabla u) - \dot{\mathbb{E}}_n(I(a) \mathcal{A} u_L) \\ &\quad - \dot{\mathbb{E}}_n \nabla(G(a)) + \dot{\mathbb{E}}_n \left(\frac{1}{1+a} (H \cdot \nabla H - \frac{1}{2} \nabla |H|^2) \right), \\ Q_n(a, \bar{u}, H) &:= \dot{\mathbb{E}}_n(H \cdot \nabla u) - \dot{\mathbb{E}}_n(u \cdot \nabla H) + \nu \dot{\mathbb{E}}_n(\Delta H) - \dot{\mathbb{E}}_n(H \operatorname{div} u). \end{aligned}$$

Notice that if $1 + a_0$ is positive and bounded away from zero, then so is $1 + \dot{\mathbb{E}}_n a_0$ for sufficiently large n , and hence the initial data belongs to Ω_n . It is easy to check that the map

$$(a, \bar{u}, H) \rightarrow (F_n(a, \bar{u}, H), G_n(a, \bar{u}, H), Q_n(a, \bar{u}, H))$$

belongs to $\mathcal{C}(\mathbb{R}^+ \times \Omega_n; (\dot{L}_n^2)^{2d+1})$ and is locally Lipschitz with respect to the variable (a, \bar{u}, H) . Therefore, the system (2.20) has a unique maximal solution (a^n, \bar{u}^n, H^n) in the space $\mathcal{C}^1([0, T_n^*]; \Omega_n)$.

2.2.2. Uniform estimates of (a^n, u^n, H^n) . First, we note that (a^n, \bar{u}^n, H^n) satisfies the system:

$$\begin{cases} \partial_t a^n + \dot{\mathbb{E}}_n(u^n \cdot \nabla a^n) + \dot{\mathbb{E}}_n((1+a^n) \operatorname{div} u^n) = 0, \\ \partial_t \bar{u}^n - \dot{\mathbb{E}}_n \left(\frac{1}{1+a^n} \mathcal{A} \bar{u}^n \right) + \dot{\mathbb{E}}_n(u^n \cdot \nabla u^n) - \dot{\mathbb{E}}_n(I(a^n) \mathcal{A} u_L^n) \\ \quad + \nabla \dot{\mathbb{E}}_n(G(a^n)) - \dot{\mathbb{E}}_n \left(\frac{1}{1+a^n} (H^n \cdot \nabla H^n - \frac{1}{2} \nabla |H^n|^2) \right) = 0, \\ \partial_t H^n - \dot{\mathbb{E}}_n(H^n \cdot \nabla u^n) + \dot{\mathbb{E}}_n(u^n \cdot \nabla H^n) - \nu \dot{\mathbb{E}}_n(\Delta H^n) + \dot{\mathbb{E}}_n(H^n \operatorname{div} u^n) = 0 \end{cases}$$

with the initial data

$$(a^n, \bar{u}^n, H^n)|_{t=0} = (\dot{\mathbb{E}}_n a_0, 0, \dot{\mathbb{E}}_n H_0)(x), \quad x \in \mathbb{R}^d,$$

where $u^n := u_L^n + \bar{u}^n$. We claim that T_n^* may be bounded from below by the supremum T of all the time satisfying both (2.5) and (2.6), and that $(a^n, u^n, H^n)_{n \geq 1}$ is bounded in E_T^α .

In fact, since $\dot{\mathbb{E}}_n$ is an L^2 orthogonal projector, it has no effect on the energy estimates which are used in the proof of Proposition 2.5. Hence, the Proposition 2.5 applies to our approximate solution (a^n, u^n, H^n) . We remark that the dependence on n in the conditions (2.5) and (2.6) and in the inequalities (2.7)-(2.9) may be omitted. Now, as (a^n, \bar{u}^n, H^n) is spectrally supported in \mathcal{C}_n , the inequalities (2.7)-(2.9) ensure that it is bounded in $L_T^\infty(\dot{L}_n^2)$. Thus, the standard continuation criterion for ordinary differential equations implies that T_n^* is greater than any time T satisfying (2.5)-(2.6) and that, for all $n \geq 1$,

$$\begin{aligned} \|a^n\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})} &\leq 2A_0^\alpha + 1, \\ \|H^n\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})} + \nu \|H^n\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})} &\leq 2H_0^\alpha, \\ \|\bar{u}^n\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})} + b_* \mu \|\bar{u}^n\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})} \\ &\leq C(1 + A_0^\alpha)^2 \left(T(1 + (H_0^\alpha)^2) + \bar{\nu} U_L^\alpha(T) + (U_0^\alpha + \bar{\nu})(H_0^\alpha)^2 U_L^\alpha(T) \right. \\ &\quad \left. + (U_0^\alpha + (H_0^\alpha)^2 + 1)(H_0^\alpha)^{\frac{3}{2}} H_L^{\frac{1}{2}}(T) \right). \end{aligned}$$

In particular, $(a^n, u^n, H^n)_{n \geq 1}$ is bounded in E_T^α .

2.2.3. Time derivatives of $(a^n, \bar{u}^n, \bar{H}^n)$. In order to pass the limit in (a^n, u^n, H^n) , we need the compactness in time of $(\bar{a}^n, \bar{u}^n, \bar{H}^n)$ which can be stated as the following lemma.

Lemma 2.1. *Let $\bar{a}^n := a^n - \dot{\mathbb{E}}_n a_0$ and $\bar{H}^n := H^n - \dot{\mathbb{E}}_n H_0$. Then the sequence $(\bar{a}^n)_{n \geq 1}$ is bounded in*

$$\mathcal{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}) \cap \mathcal{C}^{\frac{1}{2}}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}),$$

the sequence $(\bar{u}^n)_{n \geq 1}$ is bounded in

$$\mathcal{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}) \cap \mathcal{C}^{\frac{1}{2}}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1} + \dot{B}_{2,1}^{\frac{d}{2}-2+\alpha}),$$

and the sequence $(\bar{H}^n)_{n \geq 1}$ is bounded in

$$\mathcal{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}) \cap \mathcal{C}^{\frac{1}{2}}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-2} \cap \dot{B}_{2,1}^{\frac{d}{2}-2+\alpha}).$$

Proof. The result for $(\bar{a}^n)_{n \geq 1}$ follows from the fact that $\bar{a}^n|_{t=0} = 0$ and that

$$\partial_t \bar{a}^n = -\dot{\mathbb{E}}_n(\operatorname{div}(u^n(1 + a^n))). \quad (2.21)$$

Indeed, as $(a^n, u^n)_{n \geq 1}$ is bounded in E_T^α , by the product law in Besov spaces, the right-hand side of (2.21) is bounded in $L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$.

For $(\bar{u}^n)_{n \geq 1}$, it suffices to prove that $(\partial_t \bar{u}^n)_{n \geq 1}$ is bounded in $L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta} + \dot{B}_{2,1}^{\frac{d}{2}-2+\beta})$ for $\beta \in \{0, \alpha\}$. We rewrite the equation for \bar{u}^n as

$$\begin{aligned} \partial_t \bar{u}^n &= -\dot{\mathbb{E}}_n(u^n \cdot \nabla u^n) + \dot{\mathbb{E}}_n\left(\frac{1}{1 + a^n} \cdot \mathcal{A} \bar{u}^n\right) - \nabla \dot{\mathbb{E}}_n(G(a)) \\ &\quad + \dot{\mathbb{E}}_n\left(\frac{1}{1 + a^n}(H^n \cdot \nabla H^n - \frac{1}{2}|H^n|^2)\right) - \dot{\mathbb{E}}_n(I(a^n \mathcal{A} u_L^n)) \end{aligned} \quad (2.22)$$

with $\bar{u}^n|_{t=0} = 0$. Because $(u^n)_{n \geq 1}$, $(\bar{u}^n)_{n \geq 1}$, and $(H^n)_{n \geq 1}$ are bounded in $L_T^2(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}) \cap L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$, and \bar{a}^n is bounded in $L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})$, we easily deduce that the first four terms on the right-hand side of (2.22) are in $L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-2} \cap \dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})$ and that the last one is in $L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$ uniformly.

Similarly, the estimate for $(\bar{H}^n)_{n \geq 1}$ follows from the facts that $\bar{H}^n|_{t=0} = 0$ and that

$$\partial_t \bar{H}^n = -\mathbb{E}_n(u^n \cdot \nabla H^n) + \mathbb{E}_n(H^n \cdot \nabla u^n) + \nu \mathbb{E}_n(\Delta H^n) - \nu \mathbb{E}_n(H^n \operatorname{div} u^n). \quad (2.23)$$

Indeed, as u^n and H^n are bounded in $L_T^2(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}) \cap L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$, we reduce that the right-hand side of (2.23) is bounded in $L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-2} \cap \dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})$. This is a simple consequence of the product and composition laws for the homogeneous Besov spaces, as stated in Appendix A. \square

2.2.4. Compactness and convergence of $(a^n, \bar{u}^n, \bar{H}^n)$. By the results obtained in the above three steps, we begin to discuss the compactness and convergence of $(a^n, \bar{u}^n, \bar{H}^n)$. The arguments are very similar to that of [1] on the isentropic Navier-Stokes equations. Here we present them for the sake of completeness. As in [1], we introduce a sequence $(\varphi_p)_{p \geq 1}$ of smooth functions with values in $[0, 1]$, supported in the ball $B(0, p+1)$ and equal to 1 on $B(0, p)$. According to the previous lemma, the sequence $(\bar{a}^n)_{n \geq 1}$ is bounded in the space $\mathcal{C}^{\frac{1}{2}}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$. Moreover, we have:

(i) By virtue of Proposition A.3, $(\varphi_p \bar{a}^n)_{n \geq 1}$ is bounded in

$$\mathcal{C}([0, T]; B_{2,1}^{\frac{d}{2}} \cap B_{2,1}^{\frac{d}{2}+\alpha}) \cap \mathcal{C}^{\frac{1}{2}}([0, T]; B_{2,1}^{\frac{d}{2}-1} \cap B_{2,1}^{\frac{d}{2}-1+\alpha});$$

(ii) According to Proposition A.4,

$$\text{the map } z \rightarrow \varphi_p z \text{ is compact from } B_{2,1}^{\frac{d}{2}+\alpha} \text{ to } B_{2,1}^{\frac{d}{2}-1+\alpha};$$

(iii) Since $\varphi_p \bar{a}^n$ is uniformly bounded in $\mathcal{C}^{\frac{1}{2}}([0, T]; B_{2,1}^{\frac{d}{2}-1+\alpha})$, we have $\varphi_p \bar{a}^n$ is uniformly equicontinuous with values in $B_{2,1}^{\frac{d}{2}-1+\alpha}$.

Therefore, the Ascoli's theorem ensues that there exists some function \bar{a}_p such that, up to a subsequence,

$$(\varphi_p \bar{a}^n)_{n \geq 1} \text{ converges to } \bar{a}_p \text{ in } \mathcal{C}^{\frac{1}{2}}([0, T]; B_{2,1}^{\frac{d}{2}-1+\alpha}).$$

Using Cantor's diagonal process, we can then find a subsequence of $(\bar{a}^n)_{n \geq 1}$ (still denoted by $(\bar{a}^n)_{n \geq 1}$) such that for, all $p \geq 1$,

$$\varphi_p \bar{a}^n \text{ converges to } \bar{a}_p \text{ in } \mathcal{C}^{\frac{1}{2}}([0, T]; B_{2,1}^{\frac{d}{2}-1+\alpha}).$$

As $\varphi_p \varphi_{p+1} = \varphi_p$, we have $\bar{a}_p = \varphi_p \bar{a}_{p+1}$. Thus, we can easily deduce that there exists some function \bar{a} such that, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\varphi \bar{a}^n \text{ tends to } \varphi \bar{a} \text{ in } \mathcal{C}^{\frac{1}{2}}([0, T]; B_{2,1}^{\frac{d}{2}-1+\alpha}).$$

A similar argument gives us that there exists a vector field \bar{u} such that (up to extraction), for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$(\varphi \bar{u}^n)_{n \geq 1} \text{ converges to } \varphi \bar{u} \text{ in } \mathcal{C}^{\frac{1}{2}}([0, T]; B_{2,1}^{\frac{d}{2}-2+\alpha}),$$

and there exists a vector field \bar{H} such that (up to extraction), for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, the sequence $(\varphi \bar{H}^n)_{n \geq 1}$ converges to $\varphi \bar{H}$ in $\mathcal{C}^{\frac{1}{2}}([0, T]; B_{2,1}^{\frac{d}{2}-2+\alpha})$.

Next, the uniform bounds supplied by the second step and the Fatou property together ensure that $1 + a$ is positive and

$$(\bar{a}, \bar{u}, \bar{H}) \in \tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha}) \times \tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}) \times \tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}).$$

We claim that (\bar{u}, \bar{H}) also belongs to $(L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha}))^{d+d}$. Indeed, since $(\bar{u}^n)_{n \geq 1}$ is bounded in $L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})$, we know that \bar{u} belongs to the set $\mathcal{M}_T(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})$ of bounded measures on $[0, T]$ with values in the space $\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha}$, and that

$$\int_0^T d\|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha}} \leq C_T,$$

where C_T stands for the right-hand side of (2.8).

It is clear that the same inequality holds for $\dot{\mathbb{E}}_n \bar{u}$, for all $n \geq 1$. As $\bar{u} \in L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$, we have $\dot{\mathbb{E}}_n \bar{u} \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})$. Thus, we may write

$$\int_0^T \|\dot{\mathbb{E}}_n \bar{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha}} dt \leq C_T \text{ for all } n \geq 1.$$

Using the definition of the norm in $\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha}$, the above inequality implies that

$$\lim_{N \rightarrow \infty} \sum_{|j| \leq N} 2^{j(\frac{d}{2}+1+\beta)} \int_0^T \|\dot{\Delta}_j \bar{u}\|_{L^2} dt \leq C_T, \quad \beta \in \{0, \alpha\}.$$

Therefore, $\bar{u} \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})$. A similar argument implies that $\bar{H} \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})$.

Interpolating the above convergence results, we may get better convergence results for $(\bar{a}^n, \bar{u}^n, \bar{H}^n)$ and pass to the limit in (2.20). Defining

$$(a, u, H) := (\bar{a} + a_0, u_L + \bar{u}, \bar{H} + H_0),$$

we thus get a solution (a, u, H) of (1.7)-(1.10) with the initial data (a_0, u_0, H_0) . Using the equations of (a, u, H) and the product laws, we also have $(\partial_t + u \cdot \nabla)a \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})$, $\partial_t u \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$, and $\partial_t H \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$. Proposition 2.2 therefore guarantees that $a \in \tilde{\mathcal{C}}_T(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})$. Obviously, we have $u \in \tilde{\mathcal{C}}_T(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$ and $H \in \tilde{\mathcal{C}}_T(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$.

Remark 2.1. According to the properties of the semigroup for the heat kernel, we have the following estimates:

$$\|u_L\|_{L_T^1(\dot{B}_{2,1}^{s+2})} \leq C \sum_{j \in \mathbb{Z}} 2^{js} (1 - e^{-\kappa \bar{\nu} 2^{2j} T}) \|\dot{\Delta}_j u_0\|_{L^2},$$

$$\|H_L\|_{L_T^1(\dot{B}_{2,1}^{s+2})} \leq C \sum_{j \in \mathbb{Z}} 2^{js} (1 - e^{-\kappa \nu 2^{2j} T}) \|\dot{\Delta}_j H_0\|_{L^2},$$

where κ is constant.

Remark 2.2. Combining (2.5) and (2.6) with Remark 2.1 yields a rather explicit lower bound on the lifespan T^* of the solution. Indeed, using the fact that

$$1 - e^{-\beta T 2^{2j}} \leq (\beta T)^{\frac{\alpha}{2}} 2^{j\alpha},$$

we may find some constant c , depending only on $d, b_*, b^*, \alpha, \lambda, \mu$, and ν , such that

$$T^* \geq \sup \left\{ T \mid T \leq \frac{c}{(1 + A_0^\alpha)^2 ((1 + (H_0^\alpha)^2 + (U_0^\alpha)^2)(H_0^\alpha)^2 + U_0^\alpha)}, \quad T \leq \frac{c}{(1 + A_0^\alpha)^{\frac{2}{\alpha}}}, \right. \\ \left. \sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{2}-1+\alpha)} (1 - e^{-\kappa \nu 2^{2j} T}) \|\dot{\Delta}_j u_0\|_{L^2} \leq \frac{c}{(1 + A_0^\alpha)^2 (1 + (1 + U_0^\alpha)(H_0^\alpha)^2)} \right\}.$$

2.3. Uniqueness of the local solution. In this subsection, we discuss the uniqueness of the local solution obtained in the previous subsection. Let (a^1, u^1, H^1) and (a^2, u^2, H^2) be two solutions in E_T^α of the system (1.7)-(1.10) with the same initial data. We assume, without loss of generality, that (a^2, u^2, H^2) is the solution constructed in the previous subsection satisfying

$$1 + \inf_{(t,x) \in [0,T] \times \mathbb{R}^d} a^2(t,x) > 0.$$

We need to prove that $(a^1, u^1, H^1) = (a^2, u^2, H^2)$ on $[0, T] \times \mathbb{R}^d$. To this end, we shall estimate

$$(\hat{a}, \hat{u}, \hat{H}) := (a^2 - a^1, u^2 - u^1, H^2 - H^1)$$

with respect to a suitable norm. A direction computation $(\hat{a}, \hat{u}, \hat{H})$ satisfies

$$\begin{cases} \partial_t \hat{a} + u^2 \cdot \nabla \hat{a} + \hat{G}_0 = 0, \\ \partial_t \hat{u} + u^2 \cdot \nabla \hat{u} + \hat{u} \cdot \nabla u^1 - \frac{1}{1+a^2} \mathcal{A} \hat{u} = \hat{G}_1 + \hat{G}_2 + \hat{G}_3, \\ \partial_t \hat{H} + u^2 \cdot \nabla \hat{H} + \hat{H} \cdot \nabla u^1 - \nu \Delta \hat{H} = \hat{G}_4, \end{cases} \quad (2.24)$$

with

$$\begin{aligned} \hat{G}_0 &:= \hat{u} \cdot \nabla a^1 + \hat{a} \operatorname{div} u^2 + (1 + a^1) \operatorname{div} \hat{u}, \\ \hat{G}_1 &:= (I(a^1) - I(a^2)) \mathcal{A} u^1, \quad \hat{G}_2 := \nabla (G(a^1) - G(a^2)), \\ \hat{G}_3 &:= \frac{1}{1+a^2} H^2 \nabla H^2 - \frac{1}{1+a^1} H^1 \nabla H^1 \\ &\quad - \frac{1}{1+a^2} \nabla |H^2|^2 + \frac{1}{1+a^1} \nabla |H^1|^2, \\ \hat{G}_4 &:= H^2 \cdot \nabla \hat{u} - \hat{u} \cdot \nabla H^1 - \operatorname{div} u^1 \cdot \hat{H} - \operatorname{div} \hat{u} \cdot H^2. \end{aligned}$$

Due to the hyperbolic structure of the mass equation, we could not avoid a loss of one derivative in the stability estimates (the term $\hat{u} \cdot \nabla a^1$ in the first equation of (2.24) can not be better than $L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$ since we only know that $\nabla a^1 \in L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha} \cap \dot{B}_{2,1}^{\frac{d}{2}-1})$). In addition, the strong coupling in the equations for $(\hat{a}, \hat{u}, \hat{H})$ implies that this loss of one

derivative also results in a loss of one derivative when bounding \hat{u} and \hat{H} . Hence, we expect to prove uniqueness in the following function space,

$$F_T^\alpha := \mathcal{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}) \times (\mathcal{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-2+\alpha}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\alpha}))^d \\ \times (\mathcal{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-2+\alpha}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\alpha}))^d.$$

First, we show that $(\hat{a}, \hat{u}, \hat{H})$ belongs to F_T^α . For \hat{a} , a similar argument to that in the proof of Lemma 2.1 implies that $\partial_t \bar{a}^i \in L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})(i = 1, 2)$. Hence, we get $\bar{a}^i \in \mathcal{C}^{\frac{1}{2}}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$. To deal with \hat{u} , we introduce $\bar{u}^i := u^i - \bar{u}_L$, where \bar{u}_L is the solution of

$$\partial_t \bar{u}_L - \mathcal{A} \bar{u}_L = -\nabla G(a_0), \quad \bar{u}_L|_{t=0} = u_0.$$

Obviously, we have $\bar{u}^i|_{t=0} = 0$ and

$$\partial_t \bar{u}^i = \mathcal{A} \bar{u}^i - I(a^i) \mathcal{A} u^i - u^i \nabla u^i + \frac{1}{a^i} \left(H^i \cdot \nabla H^i - \frac{1}{2} \nabla |H^i|^2 \right) - \nabla (G(a^i) - G(a_0)).$$

Since $\bar{a}^i \in L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$ and $(a^i, u^i, H^i) \in E_T^\alpha$, the right-hand side of the above equation belongs to $L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})$. Hence, \bar{u}^i belongs to $\mathcal{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})$. Similarly, since H^i satisfies

$$\partial_t H^i = H^i \cdot \nabla u^i - u^i \cdot \nabla H^i - \nu \Delta H^i - (\operatorname{div} u^i) H^i,$$

we easily deduce that $\partial_t H^i \in L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})$. Thus, $H^i \in \mathcal{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})$ and hence we conclude that $(\hat{a}, \hat{u}, \hat{H}) \in F_T^\alpha$.

Next, applying Proposition 2.1 to the first equation of (2.24), we get, for all $\bar{T} \in [0, T]$, that

$$\|\hat{a}\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \leq \exp \left\{ C \|u^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \right\} \|\hat{G}_0\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})}.$$

By Proposition A.2, an easy computation gives

$$\|\hat{G}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}} \leq C \left\{ \|\hat{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla a^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}} + \|\operatorname{div} u^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\hat{a}\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha}} \right. \\ \left. + \left(1 + \|a^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \|\hat{u}\|_{\dot{B}_{2,1}^{\frac{d}{2}+\alpha}} \right\}.$$

Hence, using Gronwall's lemma and interpolation implies that there exists a constant C_T , independent of \bar{T} , such that

$$\|\hat{a}\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \leq C_T \left(\|\hat{u}\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})} + \|\hat{u}\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})} \right). \quad (2.25)$$

Similarly, applying Proposition 2.4 to the third equation of (2.24) gives, for all $\bar{T} \in [0, T]$, that

$$\|\hat{H}\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})} + \|\hat{H}\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})} \\ \leq C \exp \left\{ C (\|u^1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|u^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}) \right\} \|\hat{G}_4\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})}.$$

By Proposition A.2 and Lemma A.2, we have

$$\begin{aligned} \|\hat{G}_4\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} &\leq C \left(\|H^2\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-1}_{2,1})} \|\hat{u}\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+\alpha}_{2,1})} \right. \\ &\quad + \|H^1\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+1}_{2,1})} \|\hat{u}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} \\ &\quad \left. + \int_0^T \|u^1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} \|\hat{H}\|_{\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1}} dt \right). \end{aligned}$$

Once again, using Gronwall's lemma and interpolation, we obtain that there exists some constant C_T , independent of \bar{T} , such that

$$\|\hat{H}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} + \|\hat{H}\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+\alpha}_{2,1})} \leq C_T \left(\|\hat{u}\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+\alpha}_{2,1})} + \|\hat{u}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} \right). \quad (2.26)$$

Similarly, applying Proposition 2.3 to the second equation of (2.24) gives

$$\begin{aligned} &\|\hat{u}\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+\alpha}_{2,1})} + \|\hat{u}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} \\ &\leq C \exp \left\{ C \int_0^{\bar{T}} (\|u^1\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|u^2\|_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \|a^2\|_{\dot{B}^{\frac{d}{2}+\alpha}_{2,1}}^{\frac{2}{\alpha}}) dt \right\} \sum_{i=1}^3 \|\hat{G}_i\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})}. \end{aligned}$$

Because $\dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d) \hookrightarrow \mathcal{C}(\mathbb{R}^d)$, we have $a^1 \in \mathcal{C}([0, T] \times \mathbb{R}^d)$. Hence, for sufficiently small \bar{T} , a^1 also satisfies (2.7). Therefore, applying Proposition A.2 and Lemmas A.2 and A.3 yields

$$\begin{aligned} \|\hat{G}_1\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} &\leq C(1 + \|a^1\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}}_{2,1})} + \|a^2\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}}_{2,1})}) \\ &\quad \times \|\hat{a}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-1+\alpha}_{2,1})} \|u^1\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+1+\alpha}_{2,1})}, \\ \|\hat{G}_2\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} &\leq C\bar{T}(1 + \|a^1\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}}_{2,1})} + \|a^2\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}}_{2,1})}) \|\hat{a}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-1+\alpha}_{2,1})}, \\ \|\hat{G}_3\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} &\leq C(1 + \|a^1\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}}_{2,1})} + \|a^2\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}}_{2,1})}) \\ &\quad \times \left\{ \left(\sum_{i=1}^2 \|H^i\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-1}_{2,1})} \|H^i\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+1}_{2,1})} \right) \|\hat{a}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-1+\alpha}_{2,1})} \right. \\ &\quad + \left(\|H^1\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-1}_{2,1})}^{\frac{1}{2}} \|H^1\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+1}_{2,1})}^{\frac{1}{2}} + \|H^2\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+1}_{2,1})} \right) \\ &\quad \left. \times \left(\|\hat{H}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} + \|\hat{H}\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+\alpha}_{2,1})} \right) \right\}. \end{aligned}$$

Therefore, there exists a constant C_T , independent of \bar{T} , such that

$$\begin{aligned} &\|\hat{u}\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+\alpha}_{2,1})} + \|\hat{u}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-2+\alpha}_{2,1})} \\ &\leq C_T \left\{ \left(\bar{T} + \|u^1\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \sum_{i=1}^2 \|H^i\|_{L^1_{\bar{T}}(\dot{B}^{\frac{d}{2}+\alpha}_{2,1})} \right) \|\hat{a}\|_{L^\infty_{\bar{T}}(\dot{B}^{\frac{d}{2}-1+\alpha}_{2,1})} \right. \end{aligned}$$

$$+ (\|H^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|H^1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^{\frac{1}{2}})(\|\hat{H}\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-2+\alpha})} + \|\hat{H}\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})}) \Big\}.$$

Note that the factors

$$\bar{T} + \|u^1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \sum_{i=1}^2 \|H^i\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})} \quad \text{and} \quad \|H^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} + \|H^1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}^{\frac{1}{2}}$$

decay to 0 when \bar{T} goes to zero. Hence, plugging the inequalities (2.25) and (2.26) into the above inequality, we conclude that $(\hat{a}, \hat{u}, \hat{H}) \equiv 0$ on a small time interval $[0, \bar{T}]$. In order to show that $\bar{T} = T$, we introduce the set

$$I := \{t \in [0, T] \mid (a^1, u^1, H^1) \equiv (a^2, u^2, H^2) \text{ on } [0, t]\}.$$

Obviously, I is a nonempty closed subset of $[0, T]$. In addition, the above arguments may be carried over to any $t \in I \cap [0, T)$, which ensures that I is also an open subset of $[0, T]$. Therefore, $I \equiv [0, T]$.

The proof of Theorem 1.1 is now completed.

2.4. A continuation criterion.

Proposition 2.6. *Under the hypotheses of Theorem 1.1, assume that the system (1.7)-(1.10) has a solution (a, u, H) on $[0, T) \times \mathbb{R}^d$ which belongs to $E_{T'}^\alpha$ for all $T' < T$ and satisfies*

$$a \in L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}+\alpha} \cap \dot{B}_{2,1}^{\frac{d}{2}}), \quad 1 + \inf_{(t,x) \in [0,T) \times \mathbb{R}^d} a(t,x) > 0;$$

$$\int_0^T \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt < \infty, \quad \int_0^T \|\nabla H\|_{\dot{B}_{2,1}^{\frac{d}{2}}} dt < \infty.$$

Then there exists a $T^ > T$ such that (a, u, H) may be extended to a solution of (1.4)-(1.10) on $[0, T^*] \times \mathbb{R}^d$ which belongs to $E_{T^*}^\alpha$.*

Proof. Note that (u, H) satisfies

$$\begin{cases} \partial_t u + u \cdot \nabla u - \frac{1}{1+a} \mathcal{A}u + \nabla G(a) = \frac{1}{1+a} (H \cdot \nabla H - \frac{1}{2} \nabla |H|^2), & u|_{t=0} = u_0, \\ \partial_t H + u \cdot \nabla H - H \cdot \nabla u - \nu \Delta H = -(\operatorname{div} u)H, & H|_{t=0} = H_0. \end{cases}$$

By taking the same arguments as those used in the proof of Proposition 2.4, we easily see that there exists a universal constant κ such that

$$\begin{aligned} & \|u\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} + \kappa b_* \mu \|u\|_{\tilde{L}_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} \\ & \leq \|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} + \|a\|_{\tilde{L}_t^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \\ & \quad + \int_0^t (1 + \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|H\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|H\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} d\tau \\ & \quad + C \int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} d\tau + C \int_0^t \|c\|_{\dot{B}_{2,1}^{\frac{2}{\alpha}+\alpha}}^{\frac{2}{\alpha}} \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} d\tau, \\ & \|H\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} + \kappa \nu \|H\|_{\tilde{L}_t^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} \end{aligned}$$

$$\begin{aligned} &\leq \|H_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} + \|(\operatorname{div} u)H\|_{\tilde{L}_t^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\ &\quad + C \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|H\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} d\tau, \end{aligned}$$

with $c = -I(a)$, $\beta \in \{0, \alpha\}$. Adding the above two inequalities and applying Gronwall's inequality, we then obtain, for all $\beta \in \{0, \alpha\}$ and $T' < T$, that

$$\begin{aligned} &\|u\|_{\tilde{L}_{T'}^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} + \mu \|u\|_{\tilde{L}_{T'}^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} + \|H\|_{\tilde{L}_{T'}^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} + \nu \|H\|_{\tilde{L}_{T'}^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} \\ &\leq C \left(\|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} + \|H_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} + \|a\|_{\tilde{L}_{T'}^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \right) \\ &\quad \times \exp \left\{ C \left(\int_0^{T'} \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + (1 + \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|\nabla H\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|a\|_{\dot{B}_{2,1}^{\frac{d}{2}+\alpha}}^{\frac{2}{\alpha}} \right) dt \right\} \end{aligned}$$

for some constant C depending only on d, α , and the viscosity coefficients. Hence, (u, H) is bounded in $\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})$.

Replacing $\|\dot{\Delta}_j u_0\|_{L^2}$ and $\|\dot{\Delta}_j H_0\|_{L^2}$ by $\|\dot{\Delta}_j u\|_{L_T^\infty(L^2)}$ and $\|\dot{\Delta}_j H\|_{L_T^\infty(L^2)}$ in Remark 2.2, respectively, we get an $\epsilon > 0$ such that, for any $T' \in [0, T]$, the system (1.7)-(1.10) with initial data $(a(T'), u(T'), H(T'))$ has a solution for $t \in [0, \epsilon]$. Taking $T' = T - \epsilon/2$ and using the fact that the solution (a, u, H) is unique on $[0, T]$, we thus get a continuation of (a, u, H) beyond T . \square

3. LOW MACH NUMBER LIMIT FOR THE COMPRESSIBLE MHD EQUATIONS

In this section we shall study the low Mach number limit of the compressible MHD equations (1.19)-(1.21) for the local solution obtained in Theorem 1.1. The main strategy is to apply the Leray projector on the system to divide it into the incompressible part and acoustic part and then estimate the acoustic part and the difference of the incompressible part with the incompressible MHD equations. We shall follow and adapt some ideas developed by Danchin [10] on the isentropic Navier-Stokes equations. Before we begin our proof, we briefly describe the process as follows. Firstly, we use the dispersive inequalities of linear wave equations to bound a suitable norm of $(b^\epsilon, \mathcal{P}^\perp u^\epsilon)$, and this bound will be controlled by the norm of (b^ϵ, u^ϵ) in $E_{\epsilon, T}^{\frac{d}{2}+\beta}$ times some positive power of ϵ . Secondly, by means of estimates for the non-stationary incompressible MHD equations (see Proposition B.3) and paradifferential calculus, we get a priori bounds for $\epsilon^{-\beta}(\mathcal{P}u^\epsilon - v, H^\epsilon - B)$ in term of $\|(b^\epsilon, u^\epsilon, H^\epsilon)\|_{E_{\epsilon, T}^{\frac{d}{2}+\beta}}$ and $\|(v, B)\|_{F_T^{\frac{d}{2}+\beta}}$. Thirdly, we show uniform bounds for $\|(b^\epsilon, u^\epsilon, H^\epsilon)\|_{E_{\epsilon, T}^{\frac{d}{2}+\beta}}$ in term of (v, B) and the initial data. We then use a bootstrap argument to close the estimates on the first three steps. Finally, we use a continuity argument to complete our proof.

Proof of Theorem 1.2. Throughout the proof we shall use the following notations:

$$w^\epsilon := \mathcal{P}u^\epsilon - v, \quad B^\epsilon := H^\epsilon - B,$$

$$\begin{aligned}
X_\beta(T) &:= \|b^\epsilon\|_{L_T^1(\dot{B}_\epsilon^{\frac{d}{2}+\beta,1})} + \|b^\epsilon\|_{L_T^\infty(\dot{B}_\epsilon^{\frac{d}{2}+\beta,\infty})} \\
&\quad + \|\mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} + \|\mathcal{P}^\perp u^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})}, \\
V_\beta(T) &:= \|v\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} + \|v\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\
&\quad + \|B\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} + \|B\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})}, \\
W_\beta(T) &:= \|w^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} + \|w^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\
&\quad + \|B^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} + \|B^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})}, \\
Y_\beta^p(T) &:= \|b^\epsilon\|_{L_T^p(\dot{B}_{\infty,1}^{\beta-1+\frac{1}{p}})} + \|\mathcal{P}^\perp u^\epsilon\|_{L_T^p(\dot{B}_{\infty,1}^{\beta-1+\frac{1}{p}})} \quad \text{if } d=3 \text{ and } 2 < p < \infty, \\
Y_\beta(T) &:= \|b^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\beta-\frac{3}{4}})} + \|\mathcal{P}^\perp u^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\beta-\frac{3}{4}})} \quad \text{if } d=2.
\end{aligned}$$

We shall also use the notations $P_\beta(T) := V_\beta(T) + W_\beta(T)$ and

$$X_0^\beta := \|b_0\|_{\dot{B}_\epsilon^{\frac{d}{2}+\beta,\infty}} + \|\mathcal{P}^\perp u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} + \|H_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}}.$$

In our arguments below the time T will sometimes be omitted and β will always stand for 0 or α .

3.1. Dispersive estimates for $(b^\epsilon, \mathcal{P}^\perp u^\epsilon)$. We first recall the dispersive inequalities for the following (reduced) system of acoustics:

$$\begin{cases} \partial_t b + \epsilon^{-1} \Lambda \Psi = F, \\ \partial_t \Psi - \epsilon^{-1} \Lambda b = G, \\ (b, \Psi)|_{t=0} = (b_0, \Psi_0)(x), \quad x \in \mathbb{R}^d. \end{cases} \quad (3.1)$$

Recall that Λ is defined as $\Lambda := \sqrt{-\Delta}$ in Appendix A.

Proposition 3.1 ([10]). *Let (b, Ψ) be a solution of (3.1). Then, for any $s \in \mathbb{R}$ and positive T (possibly infinite), the following estimate holds:*

$$\|(b, \Psi)\|_{\tilde{L}_T^r(\dot{B}_{p,1}^{s+d(\frac{1}{p}-\frac{1}{2})\frac{1}{r}})} \leq C\epsilon^{\frac{1}{r}}\|(b_0, \Psi_0)\|_{\dot{B}_{2,1}^s} + \epsilon^{1+\frac{1}{r}-\frac{1}{\bar{r}'}}\|(F, G)\|_{\tilde{L}_T^{\bar{r}'}(\dot{B}^{s+d(\frac{1}{\bar{r}'}-\frac{1}{2})+\frac{1}{\bar{r}}})}$$

with

$$\begin{aligned}
p &\geq 2, \quad \frac{2}{r} \leq \min\{1, \gamma(p)\}, \quad (r, p, d) \neq (2, \infty, 3), \\
\bar{p} &\geq 2, \quad \frac{2}{\bar{r}} \leq \min\{1, \gamma(\bar{p})\}, \quad (\bar{r}, \bar{p}, d) \neq (2, \infty, 3),
\end{aligned}$$

where

$$\gamma(q) := (d-1)\left(\frac{1}{2} - \frac{1}{q}\right), \quad \frac{1}{\bar{p}} + \frac{1}{\bar{p}'} = 1, \quad \frac{1}{\bar{r}} + \frac{1}{\bar{r}'} = 1.$$

It is easy to check that $(b^\epsilon, \mathcal{P}^\perp u^\epsilon)$ satisfy the system:

$$\begin{cases} \partial_t b^\epsilon + \epsilon^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon = F^\epsilon, \\ \partial_t \mathcal{P}^\perp u^\epsilon + \epsilon^{-1} \nabla b^\epsilon = G^\epsilon, \end{cases} \quad (3.2)$$

with $F^\epsilon := -\operatorname{div}(b^\epsilon u^\epsilon)$ and

$$G^\epsilon := -\mathcal{P}^\perp \left(u^\epsilon \cdot \nabla u^\epsilon + \frac{\mathcal{A}u^\epsilon}{1 + \epsilon b^\epsilon} + \frac{K(\epsilon b^\epsilon)}{\epsilon} \nabla b^\epsilon + \frac{1}{1 + \epsilon b^\epsilon} (H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla(|H^\epsilon|^2)) \right),$$

where $K(z) := \frac{P'(1+z)}{1+z} - 1$ (hence $K(0) = 0$). Obviously, the dispersive estimates stated in Proposition 3.1 are also true for the system (3.2) since b^ϵ and $d^\epsilon := \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon$ satisfy (3.1) with source terms F^ϵ and $\Lambda^{-1} \operatorname{div} G^\epsilon$, and $\Lambda^{-1} \operatorname{div}$ is a homogeneous multiplier of degree 0. Hence, we have, for $d = 3$ and $2 < p < +\infty$,

$$Y_\alpha^p \leq C \epsilon^{\frac{1}{p}} \left(\|(b_0, \mathcal{P}u_0)\|_{\dot{B}_{2,1}^{\frac{1}{2}+\alpha}} + \|(F^\epsilon, \Lambda^{-1} \operatorname{div} G^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{1}{2}+\alpha})} \right). \quad (3.3)$$

and for $d = 2$,

$$Y_\alpha \leq C \epsilon^{\frac{1}{4}} \left(\|(b_0, \mathcal{P}u_0)\|_{\dot{B}_{2,1}^\alpha} + \|(F^\epsilon, \Lambda^{-1} \operatorname{div} G^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^\alpha)} \right). \quad (3.4)$$

From Propositions A.1 and A.2, we easily conclude that, for $d = 3$ or $d = 2$,

$$\begin{aligned} \|\operatorname{div}(b^\epsilon u^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} &\leq C \left(\|b^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|u^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})} \right. \\ &\quad \left. + \|b^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})} \|u^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \right) \\ &\leq C(X_0(X_\alpha + P_\alpha) + X_\alpha(X_0 + P_0)), \\ \|\mathcal{P}^\perp(u^\epsilon \cdot \nabla u^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} &\leq C \|u^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla u^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \\ &\leq C(X_0 + P_0)(X_\alpha + P_\alpha), \\ \|\mathcal{P}^\perp(I(\epsilon b^\epsilon) \cdot \mathcal{A}u^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} &\leq C \epsilon \|b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\mathcal{A}u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \\ &\leq C X_0(X_\alpha + P_\alpha), \\ \|K(\epsilon b^\epsilon) \nabla b^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} &\leq C \epsilon \|b^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla b^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \\ &\leq C \epsilon X_0 X_\alpha. \end{aligned}$$

Since

$$\begin{aligned} &\mathcal{P}^\perp \left(\frac{1}{1 + \epsilon b^\epsilon} (H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla(|H^\epsilon|^2)) \right) \\ &= \mathcal{P}^\perp \left(H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla(|H^\epsilon|^2) \right) + \mathcal{P}^\perp \left(I(\epsilon b^\epsilon) (H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla(|H^\epsilon|^2)) \right), \end{aligned}$$

we have

$$\begin{aligned} \|\mathcal{P}^\perp(H^\epsilon \cdot \nabla H^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} &\leq C \|H^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla H^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \\ &\leq C P_0 P_\alpha, \\ \|\mathcal{P}^\perp(\nabla(|H^\epsilon|^2))\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} &\leq C P_0 P_\alpha, \\ \|\mathcal{P}^\perp(I(\epsilon b^\epsilon) H^\epsilon \cdot \nabla H^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} &\leq C \epsilon \|b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|H^\epsilon \cdot \nabla H^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \\ &\leq C X_0 P_0 P_\alpha, \end{aligned}$$

$$\|\mathcal{P}^\perp \left(I(\epsilon b^\epsilon) |\nabla H^\epsilon|^2 \right)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})} \leq C X_0 P_0 P_\alpha.$$

Plugging the above inequalities into (3.3) or (3.4), we conclude that for $d = 3$ and $2 < p < \infty$,

$$Y_\alpha^p \leq C \epsilon^{\frac{1}{p}} (X_\alpha^0 + X_\alpha + (X_0 + P_0)(X_\alpha + P_\alpha) + X_0 P_0 P_\alpha), \quad (3.5)$$

and for $d = 2$,

$$Y_\alpha \leq C \epsilon^{\frac{1}{4}} (X_\alpha^0 + X_\alpha + (X_0 + P_0)(X_\alpha + P_\alpha) + X_0 P_0 P_\alpha). \quad (3.6)$$

3.2. Estimates for (w^ϵ, B^ϵ) . From the system (1.20)-(1.21) and (1.15)-(1.16), a direct computation gives

$$\begin{cases} \partial_t w^\epsilon - \mu \Delta w^\epsilon + \mathcal{P}(A^\epsilon \cdot \nabla w^\epsilon) + \mathcal{P}(w^\epsilon \cdot \nabla A^\epsilon) - \mathcal{P}(B^\epsilon \cdot \nabla B) \\ \quad - \mathcal{P}(B \cdot \nabla B^\epsilon) = -\mathcal{P}M^\epsilon + \mathcal{P}L^\epsilon, \\ \partial_t B^\epsilon - \nu \Delta B^\epsilon + \mathcal{P}(A^\epsilon \cdot \nabla B^\epsilon) - \mathcal{P}(B^\epsilon \cdot \nabla A^\epsilon) + \mathcal{P}(w^\epsilon \cdot \nabla B) \\ \quad - \mathcal{P}(B \cdot \nabla w^\epsilon) = -\mathcal{P}Q^\epsilon, \\ (w^\epsilon, B^\epsilon)|_{t=0} = (0, 0), \end{cases}$$

where

$$A^\epsilon := \mathcal{P}^\perp u^\epsilon + v,$$

$$M^\epsilon := \mathcal{P}^\perp u^\epsilon \cdot \nabla v + v \cdot \nabla \mathcal{P}^\perp u^\epsilon + w^\epsilon \cdot \nabla w^\epsilon + I(\epsilon b^\epsilon) \mathcal{A} u^\epsilon,$$

$$L^\epsilon := B^\epsilon \cdot \nabla B^\epsilon + I(\epsilon b^\epsilon) (H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} |\nabla H^\epsilon|^2),$$

$$Q^\epsilon := B^\epsilon \cdot \nabla w^\epsilon - w^\epsilon \cdot \nabla B^\epsilon + B \cdot \nabla \mathcal{P}^\perp u^\epsilon - \mathcal{P}^\perp u^\epsilon \cdot \nabla B - (\operatorname{div} \mathcal{P}^\perp u^\epsilon) H^\epsilon.$$

Applying Proposition B.3 with $s = d/2 - 1 + \beta$ yields

$$\begin{aligned} W_\beta &\leq C \left(\|M^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} + \|L^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} + \|Q^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \right) \\ &\quad \times \exp \left\{ C \int_0^T (\|\nabla A^\epsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla B\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) dt \right\} \\ &\leq C e^{C(V_0+X_0)} \left(\|M^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} + \|L^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} + \|Q^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \right). \end{aligned} \quad (3.7)$$

We now bound M^ϵ , L^ϵ , and Q^ϵ . First, we readily have

$$\|w^\epsilon \cdot \nabla w^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C \|w^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla w^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C W_0 W_\beta, \quad (3.8)$$

$$\|B^\epsilon \cdot \nabla B^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C \|B^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla B^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C W_0 W_\beta, \quad (3.9)$$

$$\|B^\epsilon \cdot \nabla w^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C \|B^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla w^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C W_0 W_\beta, \quad (3.10)$$

$$\|w^\epsilon \cdot \nabla B^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C \|w^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla B^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C W_0 W_\beta. \quad (3.11)$$

Next, by interpolation and (ii) in Remark A.2, we have

$$\|b^\epsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \|b^\epsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+\alpha-1}}^\alpha \|b^\epsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+\alpha}}^{1-\alpha} \leq \epsilon^{\alpha-1} \|b^\epsilon\|_{\dot{B}_\epsilon^{\frac{d}{2}+\alpha,\infty}}, \quad (3.12)$$

$$\begin{aligned}
& \|I(\epsilon b^\epsilon) \mathcal{A} u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\
& \leq C\epsilon \|b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\mathcal{A} u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\
& \leq C\epsilon^\alpha \|b^\epsilon\|_{L_T^\infty(\tilde{B}_\epsilon^{\frac{d}{2}+\alpha,\infty})} \|u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} \\
& \leq C\epsilon^\alpha X_\alpha(V_\beta + W_\beta + X_\beta), \tag{3.13}
\end{aligned}$$

$$\begin{aligned}
& \|I(\epsilon b^\epsilon) H^\epsilon \cdot \nabla H^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\
& \leq C\epsilon \|b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|H^\epsilon \cdot \nabla H^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\
& \leq C\epsilon^\alpha \|b^\epsilon\|_{L_T^\infty(\tilde{B}_\epsilon^{\frac{d}{2}+\alpha,\infty})} \|H^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla H^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\
& \leq C\epsilon^\alpha X_\alpha(W_0 + V_0)(W_\beta + V_\beta), \tag{3.14}
\end{aligned}$$

$$\|I(\epsilon b^\epsilon) \cdot \nabla |H^\epsilon|^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C\epsilon^\alpha X_\alpha(W_0 + V_0)(W_\beta + V_\beta). \tag{3.15}$$

We will deal with the other terms in M^ϵ , L^ϵ , and Q^ϵ according to $d = 3$ or $d = 2$. We first consider the case: $d = 3$ and $p_\alpha = 1 + \frac{1}{2\alpha}$. Since $\mathcal{P}^\perp u^\epsilon$ is small in $L^2([0, T]; \dot{B}_{\infty,1}^0)$, using interpolation and embedding, we have

$$\begin{aligned}
\|\mathcal{P}^\perp u^\epsilon\|_{L_T^2(\dot{B}_{\infty,1}^0)} & \leq \|\mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{\infty,1}^{1+\alpha})}^{\frac{1}{2}-\alpha} \|\mathcal{P}^\perp u^\epsilon\|_{L_T^{(1+2\alpha)/2\alpha}(\dot{B}_{\infty,1}^{\alpha-1/(1+2\alpha)})}^{\frac{1}{2}+\alpha} \\
& \leq C \|\mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{5/2+\alpha})}^{\frac{1}{2}-\alpha} \|\mathcal{P}^\perp u^\epsilon\|_{L_T^{(1+2\alpha)/2\alpha}(\dot{B}_{\infty,1}^{\alpha-1/(1+2\alpha)})}^{\frac{1}{2}+\alpha} \\
& \leq C\epsilon^\alpha (X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha}). \tag{3.16}
\end{aligned}$$

From (3.16) we expect to gain some smallness for $\mathcal{P}^\perp u^\epsilon \cdot \nabla v$, $v \cdot \nabla \mathcal{P}^\perp u^\epsilon$, $\mathcal{P}^\perp u^\epsilon \cdot \nabla B$, $B \cdot \nabla \mathcal{P}^\perp u^\epsilon$, and $(\operatorname{div} \mathcal{P}^\perp u^\epsilon) H^\epsilon$, by means of a judicious application of paradifferential calculus. For $\mathcal{P}^\perp u^\epsilon \cdot \nabla v$, we shall use the following decomposition (with $\eta < 1$ to be fixed hereafter):

$$\mathcal{P}^\perp u^\epsilon \cdot \nabla v = \underbrace{\sum_{q \in \mathbb{Z}} \dot{\Delta}_q \mathcal{P}^\perp u^\epsilon \cdot S_{q-1+[\log_2 \eta]} \nabla v}_{T_1} + \underbrace{\sum_{q \in \mathbb{Z}} \dot{\Delta}_q \nabla v \cdot S_{q+2-[\log_2 \eta]} \mathcal{P}^\perp u^\epsilon}_{T_2},$$

which may be seen as a slight modification of Bony decomposition.

Recall that, for any $k \in \mathbb{Z}$, we have

$$\|\dot{S}_j \nabla v\|_{L^\infty} \leq C 2^{2j} \|\nabla v\|_{\dot{B}_{\infty,1}^{-2}}.$$

Therefore,

$$\begin{aligned}
& \|\dot{\Delta}_q \mathcal{P}^\perp u^\epsilon \cdot S_{q-1+[\log_2 \eta]} \nabla v\|_{L^2} \\
& \leq C \|S_{q-1+[\log_2 \eta]} \nabla v\|_{L^\infty} \|\dot{\Delta}_q \mathcal{P}^\perp u^\epsilon\|_{L^2} \\
& \leq C \eta^2 2^{-q(\frac{d}{2}+\beta-1)} \|\nabla v\|_{\dot{B}_{\infty,1}^{-2}} (2^{q(\frac{d}{2}+\beta+1)} \|\dot{\Delta}_q \mathcal{P}^\perp u^\epsilon\|_{L^2}).
\end{aligned}$$

As the function $\dot{\Delta}_q \mathcal{P}^\perp u^\epsilon \cdot S_{q-1+[\log_2 \eta]} \nabla v \|_{L^2}$ is spectrally supported in dyadic annuli $2^q \mathcal{C}(0, R_1, R_2)$ with R_1 and R_2 independent of η , Lemma A.4 yields

$$\|T_1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} \leq C\eta^2 \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}-2}} \|\mathcal{P}^\perp u^\epsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}+1+\beta}}. \quad (3.17)$$

Next, according to the properties of quasi-orthogonality of the dyadic decomposition, we have, for all $k \in \mathbb{Z}$,

$$\dot{\Delta}_k T_2 = \sum_{q \geq k-2+[\log_2 \eta]} \dot{\Delta}_k (\dot{S}_{q+2-[\log_2 \eta]} \mathcal{P}^\perp u^\epsilon \cdot \dot{\Delta}_q \nabla v).$$

Therefore,

$$\begin{aligned} 2^{k(\frac{d}{2}+\beta-1)} \|\dot{\Delta}_k T_2\|_{L^2} &\leq C \|\mathcal{P}^\perp u^\epsilon\|_{L^\infty} \sum_{q \geq k-2+[\log_2 \eta]} 2^{(k-q)(\frac{d}{2}+\beta-1)} 2^{q(\frac{d}{2}+\beta-1)} \|\dot{\Delta}_q \nabla v\|_{L^2} \\ &\leq C\eta^{1-\beta-\frac{d}{2}} \|\mathcal{P}^\perp u^\epsilon\|_{L^\infty} \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}+\beta-1}}, \end{aligned}$$

from which it follows that

$$\|T_2\|_{\dot{B}_{2,1}^{\frac{d}{2}+\beta-1}} \leq C\eta^{1-\beta-\frac{d}{2}} \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+\beta}} \|\mathcal{P}^\perp u^\epsilon\|_{L^\infty}. \quad (3.18)$$

By (3.17), (3.18), and Hölder's inequality, we thus get

$$\begin{aligned} \|\mathcal{P}^\perp u^\epsilon \cdot \nabla v\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta-1})} &\leq C \left(\eta^2 \|v\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \|\mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} \right. \\ &\quad \left. + \eta^{1-\beta-\frac{d}{2}} \|v\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \|\mathcal{P}^\perp u^\epsilon\|_{L_T^2(L^\infty)} \right). \end{aligned}$$

Since $\dot{B}_{\infty,1}^0 \hookrightarrow L^\infty$, by choosing $\eta = \epsilon^{\frac{2\alpha}{2+d+2\beta}}$ and using (3.16), we can now conclude that

$$\|\mathcal{P}^\perp u^\epsilon \cdot \nabla v\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta-1})} \leq C\epsilon^{\frac{4\alpha}{2+d+2\beta}} (V_0 X_\beta + V_\beta (X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha})). \quad (3.19)$$

Similarly,

$$\|\mathcal{P}^\perp u^\epsilon \cdot \nabla B\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta-1})} \leq C\epsilon^{\frac{4\alpha}{2+d+2\beta}} (V_0 X_\beta + V_\beta (X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha})). \quad (3.20)$$

The term $v \cdot \nabla \mathcal{P}^\perp u^\epsilon$ may be treated similarly. In fact, using the decomposition

$$v \cdot \nabla \mathcal{P}^\perp u^\epsilon = \underbrace{\sum_{q \in \mathbb{Z}} \dot{\Delta}_q \nabla \mathcal{P}^\perp u^\epsilon \cdot S_{q-1+[\log_2 \eta]} v}_{\tilde{T}_1} + \underbrace{\sum_{q \in \mathbb{Z}} \dot{\Delta}_q v \cdot S_{q+2-[\log_2 \eta]} \nabla \mathcal{P}^\perp u^\epsilon}_{\tilde{T}_2}$$

and following the previous argument, we readily get

$$\begin{aligned} \|\tilde{T}_1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} &\leq C\eta \|v\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \|\nabla \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta})}, \\ \|\tilde{T}_2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} &\leq C\eta^{-\frac{d}{2}-\beta} \|v\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \|\nabla \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{-1})}. \end{aligned}$$

Choosing $\eta = \epsilon^{\frac{2\alpha}{2+d+2\beta}}$, we conclude that

$$\|v \cdot \nabla \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta-1})} \leq C\epsilon^{\frac{2\alpha}{2+d+2\beta}} (V_0 X_\beta + V_\beta(X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha})). \quad (3.21)$$

Similarly,

$$\|B \cdot \nabla \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta-1})} \leq C\epsilon^{\frac{4\alpha}{2+d+2\beta}} (V_0 X_\beta + V_\beta(X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha})). \quad (3.22)$$

To deal with the term $\operatorname{div}(\mathcal{P}^\perp u^\epsilon)H^\epsilon$, we introduce the decomposition

$$\begin{aligned} \operatorname{div}(\mathcal{P}^\perp u^\epsilon)H^\epsilon &= \underbrace{\sum_{q \in \mathbb{Z}} \dot{\Delta}_q \operatorname{div} \mathcal{P}^\perp u^\epsilon \cdot S_{q-1+[\log_2 \eta]} H^\epsilon}_{\bar{T}_1} \\ &\quad + \underbrace{\sum_{q \in \mathbb{Z}} \dot{\Delta}_q H^\epsilon \cdot S_{q+2-[\log_2 \eta]} \operatorname{div} \mathcal{P}^\perp u^\epsilon}_{\bar{T}_2}. \end{aligned}$$

Following the previous argument, we readily get

$$\begin{aligned} \|\bar{T}_1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} &\leq C\eta \|H^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \|\operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta})}, \\ \|\bar{T}_2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} &\leq C\eta^{-\frac{d}{2}-\beta} \|H^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \|\operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{-1})}. \end{aligned}$$

Choosing $\eta = \epsilon^{\frac{2\alpha}{2+d+2\beta}}$, we conclude that

$$\begin{aligned} \|\operatorname{div}(\mathcal{P}^\perp u^\epsilon)H^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta-1})} &\leq C\epsilon^{\frac{2\alpha}{2+d+2\beta}} \left((W_0 + V_0)X_\beta \right. \\ &\quad \left. + (W_\beta + V_\beta)(X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha}) \right). \end{aligned} \quad (3.23)$$

Now we consider the case: $d = 2$. For $\mathcal{P}^\perp u^\epsilon$, we have the following estimate:

$$\begin{aligned} \|\mathcal{P}^\perp u^\epsilon\|_{L_T^{\frac{1}{1-3\alpha}}(\dot{B}_{2,1}^{\frac{2}{1-4\alpha}-10\alpha})} &\leq \|\mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{2+\alpha})}^{1-4\alpha} \|\mathcal{P}^\perp u^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\alpha-\frac{3}{4}})}^{4\alpha} \\ &\leq C\epsilon^\alpha (X_\alpha + \epsilon^{-\frac{1}{4}} Y_\alpha) \end{aligned} \quad (3.24)$$

with $\alpha \in (0, \frac{1}{6}]$. In this part of proof, we need the following refined Bony decomposition:

$$\begin{aligned} \mathcal{P}^\perp u^\epsilon \cdot \nabla v &= \sum_{j \in \mathbb{Z}} \dot{S}_{j-1+[\log_2 \eta]} \mathcal{P}^\perp u^\epsilon \nabla \Delta_j v \\ &\quad + \sum_{j \in \mathbb{Z}} (\dot{S}_{j-1} - \dot{S}_{j-1+[\log_2 \eta]}) \mathcal{P}^\perp u^\epsilon \nabla \Delta_j v \\ &\quad + \dot{R}(\mathcal{P}^\perp u^\epsilon, \nabla v) + \dot{T}_{\nabla v} \mathcal{P}^\perp u^\epsilon. \end{aligned} \quad (3.25)$$

As in the proof of case $d = 3$ and (3.24), we have

$$\left\| \sum_{j \in \mathbb{Z}} \dot{S}_{j-1+[\log_2 \eta]} \mathcal{P}^\perp u^\epsilon \nabla \Delta_j v \right\|_{L_T^1(\dot{B}_{2,1}^\beta)}$$

$$\begin{aligned}
&\leq C\eta\|\mathcal{P}^\perp u^\epsilon\|_{L_T^\infty(\dot{B}_{\infty,1}^{-1})}\|\nabla v\|_{L_T^1(\dot{B}_{2,1}^{1+\beta})} \\
&\leq C\eta X_0 V_\beta, \\
&\left\|\sum_{j\in\mathbb{Z}}(\dot{S}_{j-1}-\dot{S}_{j-1+[\log_2 \eta]})\mathcal{P}^\perp u^\epsilon \nabla \Delta_j\right\|_{L_T^1(\dot{B}_{2,1}^\beta)} \\
&\leq C\eta^{6\alpha-1}\|\mathcal{P}^\perp u^\epsilon\|_{L_T^{\frac{1}{1-3\alpha}}(\dot{B}_{\infty,1}^{1-6\alpha})}\|\nabla v\|_{L_T^{\frac{1}{3\alpha}}(\dot{B}_{2,1}^{6\alpha-1+\beta})} \\
&\leq C\eta^{6\alpha-1}\epsilon^\alpha V_\beta(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha).
\end{aligned}$$

Choosing $\eta = \epsilon^{\alpha/(2-6\alpha)}$, we have,

$$\|\dot{T}_{\mathcal{P}^\perp u^\epsilon} \nabla v\|_{L_T^1(\dot{B}_{2,1}^\beta)} \leq C\epsilon^{\frac{\alpha}{2-6\alpha}}(X_0 V_\beta + V_\beta(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha)).$$

Using (3.24) and Remark A.3, we get

$$\begin{aligned}
\|\dot{R}(\mathcal{P}^\perp u^\epsilon, \nabla v)\|_{L_T^1(\dot{B}_{2,1}^\beta)} &\leq C\|\nabla v\|_{L_T^{\frac{1}{3\alpha}}(\dot{B}_{2,1}^{6\alpha+\beta-1})}\|\mathcal{P}^\perp u^\epsilon\|_{L_T^{\frac{1}{1-3\alpha}}(\dot{B}_{2,1}^{2-10\alpha})} \\
&\leq C\epsilon^\alpha V_\beta(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha), \\
\|\dot{S}_{q-1}\nabla v \Delta_q \mathcal{P}^\perp u^\epsilon\|_{L^2} &\leq \|\Delta_q \mathcal{P}^\perp u^\epsilon\|_{L^{\frac{2}{1-4\alpha}}} \sum_{j\leq q-2} \|\Delta_j \nabla v\|_{L^{\frac{1}{2\alpha}}} \\
&\leq 2^{-q\beta} (2^{q(2-10\alpha)} \|\Delta_q \mathcal{P}^\perp u^\epsilon\|_{L^{\frac{2}{1-4\alpha}}}) \|\nabla v\|_{\dot{B}_{\frac{1}{2\alpha},1}^{10\alpha+\beta-2}}.
\end{aligned}$$

Here we have used the facts that $\alpha, \beta \leq \frac{1}{6}$ and $10\alpha + \beta - 2 < 0$. By the embedding $\dot{B}_{2,1}^{6\alpha+\beta-1} \hookrightarrow \dot{B}_{\frac{1}{2\alpha},1}^{10\alpha+\beta-2}$, we get

$$\begin{aligned}
\|\dot{T}_{\nabla v} \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^\beta)} &\leq C\|\nabla v\|_{L_T^{\frac{1}{3\alpha}}(\dot{B}_{2,1}^{6\alpha+\beta-1})}\|\mathcal{P}^\perp u^\epsilon\|_{L_T^{\frac{1}{1-3\alpha}}(\dot{B}_{2,1}^{2-10\alpha})} \\
&\leq C\epsilon^\alpha V_\beta(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha).
\end{aligned}$$

Plugging all the above inequalities into (3.25), we finally obtain

$$\|\mathcal{P}^\perp u^\epsilon \cdot \nabla v\|_{L_T^1(\dot{B}_{2,1}^\beta)} \leq C\epsilon^{\frac{\alpha}{2-6\alpha}}(X_0 V_\beta + V_\beta(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha)). \quad (3.26)$$

Similar arguments lead to

$$\|\mathcal{P}^\perp u^\epsilon \cdot \nabla B\|_{L_T^1(\dot{B}_{2,1}^\beta)} \leq C\epsilon^{\frac{\alpha}{2-6\alpha}}(X_0 V_\beta + V_\beta(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha)), \quad (3.27)$$

$$\|v \cdot \nabla \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^\beta)} \leq C\epsilon^{\frac{\alpha}{1+6\alpha+\beta}}(V_0 X_\beta + V_\beta(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha)), \quad (3.28)$$

$$\|\operatorname{div} \mathcal{P}^\perp u^\epsilon H^\epsilon\|_{L_T^1(\dot{B}_{2,1}^\beta)} \leq C\epsilon^{\frac{\alpha}{1+6\alpha+\beta}}(P_0 X_\beta + P_\beta(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha)). \quad (3.29)$$

Plugging the estimates (3.8)-(3.10), (3.13)-(3.15), (3.19)-(3.23) or (3.26)-(3.29) in (3.7), we eventually get, if $d = 3$,

$$\begin{aligned}
W_\beta &\leq C e^{C(V_0+X_0)} \left(W_0 W_\beta + \epsilon^\alpha X_\alpha (V_\beta + (1 + W_0 + V_0)(W_\beta + V_\beta)) \right. \\
&\quad \left. + \epsilon^{\frac{2\alpha}{5+2\beta}} ((W_0 + V_0)X_\beta + (W_\beta + V_\beta)(X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha})) \right), \quad (3.30)
\end{aligned}$$

while if $d = 2$,

$$\begin{aligned} W_\beta &\leq C e^{C(V_0+X_0)} \left(W_0 W_\beta + \epsilon^\alpha X_\alpha (V_\beta + (1+W_0+V_0)(W_\beta+V_\beta)) \right. \\ &\quad \left. + \epsilon^{\frac{\alpha}{2+\beta}} (X_0 V_\beta + (W_0+V_0)X_\beta + (W_\beta+V_\beta)(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha)) \right). \end{aligned} \quad (3.31)$$

3.3. Estimates for $(b^\epsilon, \mathcal{P}^\perp u^\epsilon)$ in $E_{\epsilon,T}^{\frac{d}{2}+\beta}$. We first need the following Proposition.

Proposition 3.2 ([10]). *Let $\epsilon > 0, s \in \mathbb{R}, 1 \leq p, r < \infty$, and (a, u) be a solution of the following system:*

$$\begin{cases} \partial_t a + \dot{T}_{vj} \partial_j a + \frac{\Lambda u}{\epsilon} = F, \\ \partial_t u + \dot{T}_{vj} \partial_j u - \nu \Delta u - \frac{\Lambda a}{\epsilon} = G, \end{cases}$$

with $\dot{T}_v a := \sum_j \dot{S}_{j-1} v \dot{\Delta}_j a$. Then there exists a constant C , depending only on d, p, r , and s , such that the following estimate holds:

$$\begin{aligned} &\|a(t)\|_{\tilde{B}_\epsilon^{s,\infty}} + \|u(t)\|_{\dot{B}_{2,1}^{s-1}} + \int_0^t (\|a\|_{\tilde{B}_\epsilon^{s,1}} + \|u\|_{\dot{B}_{2,1}^{s+1}}) d\tau \\ &\leq C e^{CV_\epsilon^{p,r}(t)} \left(\|a_0\|_{\tilde{B}_\epsilon^{s,\infty}} + \|u_0\|_{\dot{B}_{2,1}^{s-1}} + \int_0^t e^{-CV_\epsilon^{p,r}(\tau)} (\|F\|_{\tilde{B}_\epsilon^{s,\infty}} + \|G\|_{\dot{B}_{2,1}^{s-1}}) d\tau \right). \end{aligned}$$

where

$$V_\epsilon^{p,r}(t) := \begin{cases} \int_0^t (\nu^{1-p} \|\nabla v\|_{\dot{B}_{\infty,\infty}^{\frac{2}{p}-2}}^p + (\epsilon^2 \nu)^{r-1} \|\nabla v\|_{L^\infty}^r) d\tau, & \text{if } p > 1, \\ \int_0^t (\|\nabla v\|_{L^\infty} + (\epsilon^2 \nu)^{r-1} \|\nabla v\|_{L^\infty}^r) d\tau, & \text{if } p = 1. \end{cases}$$

Set

$$d^\epsilon := \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon.$$

By the system (3.2), we know that (b^ϵ, d^ϵ) satisfies the following system:

$$\begin{cases} \partial_t b^\epsilon + \dot{T}_{u_j^\epsilon} \partial_j b^\epsilon + \frac{\Lambda d^\epsilon}{\epsilon} = S^\epsilon, \\ \partial_t d^\epsilon + \dot{T}_{u_j^\epsilon} \partial_j d^\epsilon - \bar{\nu} \Delta d^\epsilon - \frac{\Lambda b^\epsilon}{\epsilon} = T^\epsilon, \end{cases} \quad (3.32)$$

with

$$\begin{aligned} S^\epsilon &:= -\dot{T}'_{\partial_j b^\epsilon} u_j^\epsilon - b^\epsilon \operatorname{div} u^\epsilon, \\ T^\epsilon &:= \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp \left(K(\epsilon b^\epsilon) \frac{\nabla b^\epsilon}{\epsilon} - I(\epsilon b^\epsilon) \mathcal{A} u^\epsilon \right) \\ &\quad + \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp \left(\frac{1}{1+\epsilon b^\epsilon} (H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} |\nabla H^\epsilon|^2) \right) \\ &\quad + u^\epsilon \cdot \nabla \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon - \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp (u^\epsilon \cdot \nabla u^\epsilon) - \dot{T}'_{\partial_j \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon} u_j^\epsilon. \end{aligned}$$

Applying Proposition 3.2 to (3.32), we get

$$\begin{aligned} X_\beta(T) &\leq C e^{V_\epsilon^{p,r}(T)} \left(\|b_0\|_{\tilde{B}_\epsilon^{\frac{d}{2}+\beta,\infty}} + \|d_0\|_{\dot{B}_{2,1}^{\frac{d}{2}-1+\beta}} \right. \\ &\quad \left. + \|S^\epsilon\|_{L_T^1(\tilde{B}_\epsilon^{\frac{d}{2}+\beta,\infty})} + \|T^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \right) \end{aligned} \quad (3.33)$$

with

$$V_\epsilon^{p,r}(t) := \int_0^t (\bar{\nu}^{1-p} \|\nabla u^\epsilon(\tau)\|_{\dot{B}_{\infty,1}^{\frac{2}{p}-2}}^p + (\epsilon^2 \bar{\nu})^{r-1} \|\nabla u^\epsilon(\tau)\|_{L^\infty}^r) d\tau. \quad (3.34)$$

for any $p, r > 1$ (to be fixed hereafter). Below we give estimates on S^ϵ and T^ϵ .

3.3.1. *Estimates for S^ϵ .* Applying (ii) in Remark A.2, we have

$$\|S^\epsilon\|_{L_T^1(\dot{B}_\epsilon^{\frac{d}{2}+\beta,\infty})} \leq C \left(\|S^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} + \epsilon \|S^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \right). \quad (3.35)$$

For both $d = 3$ and $d = 2$, according to (3.12) we have the following inequality:

$$\begin{aligned} \|\dot{T}'_{\partial_j b^\epsilon} u_j^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} &\leq C \|\nabla b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \|u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} \\ &\leq C \epsilon^{\alpha-1} \|b^\epsilon\|_{L_T^\infty(\tilde{B}_\epsilon^{\frac{d}{2}+\alpha,\infty})} \|u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\ &\leq C \epsilon^{\alpha-1} X_\alpha (X_\beta + P_\beta); \end{aligned} \quad (3.36)$$

and using (ii) in Remark A.2 and (3.12), we conclude that

$$\begin{aligned} \|b^\epsilon \operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}})} &\leq C \|\operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}})} \|b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \\ &\leq C \epsilon^{\alpha-1} X_\alpha X_0, \\ \|b^\epsilon \operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})} &\leq C \|b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})} \\ &\quad + C \|\operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^{2/(2-\alpha)}(\dot{B}_{2,1}^{\frac{d}{2}})} \|b^\epsilon\|_{L_T^{2/\alpha}(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})} \\ &\leq C \epsilon^{\alpha-1} \left(\|b^\epsilon\|_{L_T^\infty(\tilde{B}_\epsilon^{\frac{d}{2}+\alpha,\infty})} \|\mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\alpha})} \right. \\ &\quad \left. + \|b^\epsilon\|_{L_T^{\frac{2}{\alpha}}(\tilde{B}_\epsilon^{\frac{d}{2}+\alpha,\frac{2}{\alpha}})} \|\mathcal{P}^\perp u^\epsilon\|_{L_T^{\frac{2}{2-\alpha}}(\dot{B}_{2,1}^{\frac{d}{2}+1})} \right) \\ &\leq C \epsilon^{\alpha-1} X_\alpha^2. \end{aligned} \quad (3.38)$$

Now we deal with the term $\|S^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})}$ in (3.35) separately for $d = 3$ or $d = 2$.

For $d = 3$, thanks to Remark A.3 we have

$$\begin{aligned} \|\dot{T}'_{\partial_j b^\epsilon} u_j^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{1}{2}+\beta})} &\leq C \|\partial_j b^\epsilon\|_{L_T^{\frac{1}{\alpha}}(\dot{B}_{\infty,1}^{2\alpha-2})} \|u_j^\epsilon\|_{L_T^{\frac{1}{1-\alpha}}(\dot{B}_{2,1}^{5/2-2\alpha+\beta})} \\ &\leq C \epsilon^\alpha (X_\alpha + P_\alpha) (\epsilon^{-\alpha} Y_\alpha^{\frac{1}{\alpha}}). \end{aligned} \quad (3.39)$$

Applying (3.16) and Proposition A.2, we get

$$\begin{aligned} \|b^\epsilon \operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{1}{2}+\beta})} &\leq C \|b^\epsilon\|_{L_T^{\frac{1}{\alpha}}(\dot{B}_{\infty,1}^{2\alpha-1})} \|\operatorname{div} \mathcal{P}^\perp u_j^\epsilon\|_{L_T^{\frac{1}{1-\alpha}}(\dot{B}_{2,1}^{3/2-2\alpha+\beta})} \\ &\quad + C \|\operatorname{div} \mathcal{P}^\perp u_j^\epsilon\|_{L_T^2(\dot{B}_{\infty,1}^{-1})} \|b^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{3}{2}+\beta})} \\ &\leq C \epsilon^\alpha X_\beta (X_\alpha + \epsilon^{-\alpha} Y_\alpha^{\frac{1}{\alpha}} + \epsilon^{-2\alpha/(1+2\alpha)} Y_\alpha^{p_\alpha}). \end{aligned} \quad (3.40)$$

Plugging (3.36)-(3.40) into (3.35), we have

$$\|S^\epsilon\|_{L_T^1(\tilde{B}_\epsilon^{\frac{3}{2}+\beta,\infty})} \leq C\epsilon^\alpha(X_\beta + P_\beta)(X_\alpha + \epsilon^{-\alpha}Y_\alpha^{\frac{1}{\alpha}} + \epsilon^{-\frac{2\alpha}{1+2\alpha}}Y_\alpha^{p_\alpha}). \quad (3.41)$$

For $d = 2$, we first remark that S^ϵ can be rewritten as

$$S^\epsilon = -\dot{T}_{\text{div } \mathcal{P}^\perp u^\epsilon} b^\epsilon - \partial_j \dot{T}'_{b^\epsilon} u_j^\epsilon.$$

By Remark A.3, (4) in Proposition A.1, and Hölder's inequality, we get

$$\begin{aligned} \|\dot{T}_{\text{div } \mathcal{P}^\perp u^\epsilon} b^\epsilon\|_{L_T^1(\dot{B}_{2,1}^\beta)} &\leq C\|\text{div } \mathcal{P}^\perp u^\epsilon\|_{L_T^{\frac{7}{4-3\alpha}}(\dot{B}_{\infty,1}^{-1})} \|b^\epsilon\|_{L_T^{\frac{7}{3+3\alpha}}(\dot{B}_{2,1}^{1+\beta})} \\ &\leq C\epsilon^{\frac{6\alpha-1}{7}} \|\text{div } \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{1+\alpha})}^{\frac{3-4\alpha}{7}} \\ &\quad \times \|\text{div } \mathcal{P}^\perp u^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\alpha-\frac{7}{4}})}^{\frac{4+4\alpha}{7}} \|b^\epsilon\|_{L_T^{\frac{7}{3+3\alpha}}(\tilde{B}_\epsilon^{1+\beta, \frac{7}{3+3\alpha}})} \\ &\leq C\epsilon^\alpha X_\beta(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha), \end{aligned} \quad (3.42)$$

$$\begin{aligned} \|\partial_j \dot{T}'_{b^\epsilon} u_j^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\beta+\alpha-\frac{1}{4}})} &\leq C\|b^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\alpha-\frac{3}{4}})} \|u^\epsilon\|_{L_T^{\frac{4}{3}}(\dot{B}_{2,1}^{\beta+\frac{3}{2}})} \\ &\leq C(X_\beta + P_\beta)Y_\alpha. \end{aligned} \quad (3.43)$$

According to the definition of hybrid Besov norms, we get the following equivalent forms:

$$\|S^\epsilon\|_{L_T^1(\tilde{B}_\epsilon^{1+\beta,\infty})} \approx \|S_{BF}^\epsilon\|_{L_T^1(\dot{B}_{2,1}^\beta)} + \epsilon\|S_{HF}^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{1+\beta})}.$$

Thus,

$$\begin{aligned} \|S^\epsilon\|_{L_T^1(\tilde{B}_\epsilon^{1+\beta,\infty})} &\leq C(\|\dot{T}_{\text{div } \mathcal{P}^\perp u^\epsilon} b^\epsilon\|_{L_T^1(\dot{B}_{2,1}^\beta)} + \|(\partial_j \dot{T}'_{b^\epsilon} u_j^\epsilon)_{BF}\|_{L_T^1(\dot{B}_{2,1}^\beta)} + \epsilon\|S_{HF}^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{1+\beta})}) \\ &\leq C(\|\dot{T}_{\text{div } \mathcal{P}^\perp u^\epsilon} b^\epsilon\|_{L_T^1(\dot{B}_{2,1}^\beta)} + \epsilon^{\alpha-\frac{1}{4}}\|\partial_j \dot{T}'_{b^\epsilon} u_j^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\beta+\alpha-\frac{1}{4}})} + \epsilon\|S^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{1+\beta})}). \end{aligned}$$

Combining (3.36)-(3.38) and (3.42)-(3.43) together, we have

$$\|S^\epsilon\|_{L_T^1(\tilde{B}_\epsilon^{1+\beta,\infty})} \leq C\epsilon^\alpha(X_\beta + P_\beta)(X_\alpha + \epsilon^{-\frac{1}{4}}Y_\alpha). \quad (3.44)$$

3.3.2. Estimates for T^ϵ . First we consider the case $d = 3$. Thanks to Proposition A.2 and Lemma A.2, we get

$$\begin{aligned} \|K(\epsilon b^\epsilon) \nabla b^\epsilon\|_{\dot{B}_{2,1}^{\frac{1}{2}+\beta}} &\leq C\|K(\epsilon b^\epsilon)\|_{L^\infty} \|\nabla b^\epsilon\|_{\dot{B}_{2,1}^{\frac{1}{2}+\beta}} + C\|K(\epsilon b^\epsilon)\|_{\dot{B}_{2,1}^{\frac{3}{2}+\beta}} \|\nabla b^\epsilon\|_{\dot{B}_{\infty,1}^{-1}} \\ &\leq C\|\epsilon b^\epsilon\|_{L^\infty} \|b^\epsilon\|_{\dot{B}_{2,1}^{\frac{3}{2}+\beta}} + C\|\epsilon b^\epsilon\|_{\dot{B}_{2,1}^{\frac{3}{2}+\beta}} \|b^\epsilon\|_{\dot{B}_{\infty,1}^0} \\ &\leq C\epsilon\|b^\epsilon\|_{\dot{B}_{2,1}^{\frac{3}{2}+\beta}} \left(\|b_{BF}^\epsilon\|_{\dot{B}_{\infty,1}^0} + \|b_{HF}^\epsilon\|_{\dot{B}_{\infty,1}^0} \right). \end{aligned}$$

Thus,

$$\begin{aligned} \left\| \frac{K(\epsilon b^\epsilon) \nabla b^\epsilon}{\epsilon} \right\|_{L_T^1(\dot{B}_{2,1}^{\frac{1}{2}+\beta})} &\leq C\|b_{BF}^\epsilon\|_{L_T^2(\dot{B}_{\infty,1}^0)} \|b^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{3/2+\beta})} \\ &\quad + C\|b^\epsilon\|_{L_T^{1/\alpha}(\dot{B}_{2,1}^{3/2+\beta})} \|b_{HF}^\epsilon\|_{L_T^{1/(1-\alpha)}(\dot{B}_{\infty,1}^0)}. \end{aligned} \quad (3.45)$$

We notice that we can replace $\mathcal{P}^\perp u^\epsilon$ by b_{BF}^ϵ in the proof of (3.16), thus we have

$$\|b_{BF}^\epsilon\|_{L_T^2(\dot{B}_{\infty,1}^0)} \leq C\epsilon^\alpha (X_\alpha + \epsilon^{-2\alpha/(1+2\alpha)} Y_\alpha^{p_\alpha}). \quad (3.46)$$

Moreover, applying (ii) in Remark A.2, we obtain that

$$\|b^\epsilon\|_{L_T^{1/\alpha}(\dot{B}_{2,1}^{3/2+\beta})} \leq C\epsilon^{2\alpha-1} \|b^\epsilon\|_{L_T^{1/\alpha}(\tilde{B}_\epsilon^{3/2+\beta,1/\alpha})}. \quad (3.47)$$

Thanks to (3) and (4) in Proposition A.1, and (ii) in Remark A.2, we obtain that

$$\begin{aligned} \|b_{HF}^\epsilon\|_{L_T^{1/(1-\alpha)}(\dot{B}_{\infty,1}^0)} &\leq C \|b_{HF}^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{3/2+\alpha})}^{(1-2\alpha)/(1-\alpha)} \|b_{HF}^\epsilon\|_{L_T^{1/\alpha}(\dot{B}_{\infty,1}^{-1+2\alpha})}^{\alpha/(1-\alpha)} \\ &\leq C (\epsilon \|b^\epsilon\|_{L_T^1(\tilde{B}_\epsilon^{3/2+\alpha,1})})^{\frac{1-2\alpha}{1-\alpha}} \epsilon^{\frac{\alpha^2}{1-\alpha}} \left(\epsilon^{-\alpha} Y_\alpha^{1/\alpha} \right)^{\frac{\alpha}{1-\alpha}} \\ &\leq C \epsilon^{1-\alpha} (X_\alpha + \epsilon^{-\alpha} Y_\alpha^{1/\alpha}). \end{aligned} \quad (3.48)$$

Plugging (3.46)-(3.48) into (3.45) gives

$$\left\| \frac{K(\epsilon b^\epsilon) \nabla b^\epsilon}{\epsilon} \right\|_{L_T^1(\dot{B}_{2,1}^{\frac{1}{2}+\beta})} \leq C \epsilon^\alpha X_\beta \left(X_\alpha + \epsilon^{-2\alpha/(1+2\alpha)} Y_\alpha^{p_\alpha} + \epsilon^{-\alpha} Y_\alpha^{1/\alpha} \right). \quad (3.49)$$

Applying Remark A.3 and (3.16), we have

$$\begin{aligned} \|\dot{T}'_{\partial_j \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon} u_j^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} &\leq C \|\partial_j \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^2(\dot{B}_{\infty,1}^{-1})} \|u^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \\ &\leq C \epsilon^\alpha (X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha}) (X_\beta + P_\beta). \end{aligned} \quad (3.50)$$

By means of Proposition A.2 and (3.12), we have

$$\begin{aligned} \|\Lambda^{-1} \operatorname{div} \mathcal{P}^\perp (I(\epsilon b^\epsilon) \mathcal{A} u^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} &\leq C \epsilon \|b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1+\beta})} \\ &\leq C \epsilon^\alpha X_\alpha (X_\beta + P_\beta). \end{aligned} \quad (3.51)$$

Now we introduce the following decomposition:

$$\begin{aligned} &u^\epsilon \cdot \nabla \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon - \Lambda^{-1} \operatorname{div} (u^\epsilon \cdot \nabla u^\epsilon) \\ &= \mathcal{P}^\perp u^\epsilon \cdot \nabla \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon - \Lambda^{-1} \operatorname{div} (\mathcal{P}^\perp u^\epsilon \cdot \nabla \mathcal{P}^\perp u^\epsilon) \\ &\quad - \Lambda^{-1} \operatorname{div} (\mathcal{P} u^\epsilon \cdot \nabla \mathcal{P} u^\epsilon) - \Lambda^{-1} \operatorname{div} (\mathcal{P}^\perp u^\epsilon \cdot \nabla \mathcal{P} u^\epsilon) \\ &\quad - \Lambda^{-1} \operatorname{div} (\mathcal{P} u^\epsilon \cdot \nabla \mathcal{P}^\perp u^\epsilon) + \mathcal{P} u^\epsilon \cdot \nabla \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon. \end{aligned} \quad (3.52)$$

Thanks to Proposition A.2, we can get the bounds for the first two terms of the right-hand side of (3.52) as the following:

$$\begin{aligned} &\|\mathcal{P}^\perp u^\epsilon \cdot \nabla \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon - \Lambda^{-1} \operatorname{div} (\mathcal{P}^\perp u^\epsilon \cdot \nabla \mathcal{P}^\perp u^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\ &\leq C \|\mathcal{P}^\perp u^\epsilon\|_{L_T^2(\dot{B}_{\infty,1}^0)} \|\mathcal{P}^\perp u^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \\ &\leq C \epsilon^\alpha X_\beta (X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha}). \end{aligned} \quad (3.53)$$

For the third term on the right-hand side of (3.52), we have

$$\begin{aligned} \|\Lambda^{-1} \operatorname{div} (\mathcal{P} u^\epsilon \cdot \nabla \mathcal{P} u^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} &\leq C \|\mathcal{P} u^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla \mathcal{P} u^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\ &\leq C P_0 P_\beta. \end{aligned} \quad (3.54)$$

Applying similar computations to those in the second step, we have,

$$\begin{aligned} & \|\Lambda^{-1} \operatorname{div} (\mathcal{P}^\perp u^\epsilon \cdot \nabla \mathcal{P} u^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\ & \leq C \epsilon^{\frac{4\alpha}{2+d+2\beta}} (P_0 X_\beta + P_\beta (X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha})), \end{aligned} \quad (3.55)$$

$$\begin{aligned} & \|\Lambda^{-1} \operatorname{div} (\mathcal{P} u^\epsilon \cdot \nabla \mathcal{P}^\perp u^\epsilon)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\ & \leq C \epsilon^{\frac{2\alpha}{2+d+2\beta}} (P_0 X_\beta + P_\beta (X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha})), \end{aligned} \quad (3.56)$$

$$\begin{aligned} & \|\mathcal{P} u^\epsilon \cdot \nabla \Lambda^{-1} \operatorname{div} \mathcal{P}^\perp u^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\ & \leq C \epsilon^{\frac{2\alpha}{2+d+2\beta}} (P_0 X_\beta + P_\beta (X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha})). \end{aligned} \quad (3.57)$$

Now, we estimate the last two terms in T^ϵ . First, we have

$$\begin{aligned} \|H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla |H^\epsilon|^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} & \leq C \|H^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|H^\epsilon\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \\ & \leq C P_0 P_\beta, \end{aligned} \quad (3.58)$$

$$\begin{aligned} \|I(\epsilon b^\epsilon) H^\epsilon \cdot \nabla H^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} & \leq C \epsilon \|b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|H^\epsilon \cdot \nabla H^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \\ & \leq C \epsilon^\alpha X_\alpha P_0 P_\beta. \end{aligned} \quad (3.59)$$

Similarly,

$$\|I(\epsilon b^\epsilon) |\nabla H^\epsilon|^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C \epsilon^\alpha X_\alpha P_0 P_\beta. \quad (3.60)$$

By (3.58)-(3.60), we obtain that

$$\left\| \frac{1}{1 + \epsilon b^\epsilon} (H^\epsilon \cdot \nabla H^\epsilon - \frac{1}{2} \nabla |H^\epsilon|^2) \right\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1+\beta})} \leq C(1 + \epsilon^\alpha X_\alpha) P_0 P_\beta. \quad (3.61)$$

Thanks to (3.49)-(3.57) and (3.61), we end up with

$$\begin{aligned} \|T^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{\frac{1}{2}+\beta})} & \leq C \left(P_0 P_\beta + \epsilon^\alpha (X_\beta + P_\beta) (X_\alpha + \epsilon^{-2\alpha/(1+2\alpha)} Y_\alpha^{p_\alpha} + \epsilon^{-\alpha} Y_\alpha^{1/\alpha}) \right. \\ & \quad \left. + \epsilon^\alpha X_\alpha P_0 P_\beta + \epsilon^{\frac{2\alpha}{5+2\beta}} (P_0 X_\beta + P_\beta (X_\alpha + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha})) \right). \end{aligned} \quad (3.62)$$

We now consider the case $d = 2$. We just have to deal with $K(\epsilon b^\epsilon) b^\epsilon$, the other terms can be treated by following the proof in the case $d = 3$. In fact, one just has to use (3.24) instead of (3.16), that is, one only needs to replace $\epsilon^{\frac{-2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha}$ by $\epsilon^{-\frac{1}{4}} Y_\alpha$. Applying Bony's decomposition for $K(\epsilon b^\epsilon) \nabla b^\epsilon$:

$$K(\epsilon b^\epsilon) \nabla b^\epsilon = \dot{T}_{\nabla b^\epsilon} K(\epsilon b^\epsilon) + \dot{R}(\nabla b^\epsilon, K(\epsilon b^\epsilon)) + \dot{T}_{K(\epsilon b^\epsilon)} \nabla b^\epsilon.$$

Thanks to Remarks A.3 and Lemma A.2, we get

$$\begin{aligned} \|\dot{T}_{\nabla b^\epsilon} K(\epsilon b^\epsilon)\|_{\dot{B}_{2,1}^\beta} & \leq C \|\nabla b^\epsilon\|_{\dot{B}_{\infty,1}^{-1}} \|K(\epsilon b^\epsilon)\|_{\dot{B}_{2,1}^{\beta+1}} \leq C \epsilon \|b^\epsilon\|_{\dot{B}_{\infty,1}^0} \|b^\epsilon\|_{\dot{B}_{2,1}^{\beta+1}}, \\ \|\dot{R}(\nabla b^\epsilon, K(\epsilon b^\epsilon))\|_{\dot{B}_{2,1}^\beta} & \leq C \|\nabla b^\epsilon\|_{\dot{B}_{2,1}^\beta} \|K(\epsilon b^\epsilon)\|_{\dot{B}_{\frac{7}{3-4\alpha},1}^{\frac{3-4\alpha}{7}}} \leq C \epsilon \|b^\epsilon\|_{\dot{B}_{2,1}^{\beta+1}} \|b^\epsilon\|_{\dot{B}_{\frac{14}{3-4\alpha},1}^{\frac{3-4\alpha}{7}}}, \\ \|\dot{T}_{K(\epsilon b^\epsilon)} \nabla b^\epsilon\|_{\dot{B}_{2,1}^\beta} & \leq C \|\nabla b^\epsilon\|_{\dot{B}_{2,1}^\beta} \|K(\epsilon b^\epsilon)\|_{L^\infty} \leq C \epsilon \|b^\epsilon\|_{\dot{B}_{2,1}^{\beta+1}} \|b^\epsilon\|_{L^\infty}. \end{aligned}$$

By means of embeddings $\dot{B}_{6/(3-4\alpha),1}^{(3-4\alpha)/3} \hookrightarrow \dot{B}_{14/(3-4\alpha),1}^{(3-4\alpha)/7} \hookrightarrow \dot{B}_{\infty,1}^0 \hookrightarrow L^\infty$, we have

$$\begin{aligned} \|K(\epsilon b^\epsilon) \nabla b^\epsilon\|_{\dot{B}_{2,1}^\beta} &\leq C\epsilon \|b^\epsilon\|_{\dot{B}_{2,1}^{\beta+1}} \|b^\epsilon\|_{\dot{B}_{\frac{14}{3-4\alpha},1}^{\frac{3-4\alpha}{7}}} \\ &\leq C\epsilon \|b^\epsilon\|_{\dot{B}_{2,1}^{\beta+1}} \left(\|b_{BF}^\epsilon\|_{\dot{B}_{\frac{14}{3-4\alpha},1}^{\frac{3-4\alpha}{7}}} + \|b_{HF}^\epsilon\|_{\dot{B}_{\frac{6}{3-4\alpha},1}^{\frac{3-4\alpha}{3}}} \right). \end{aligned}$$

Using interpolation, Hölder inequality, and (ii) in Remark A.2, we deduce that

$$\begin{aligned} &\left\| \frac{K(\epsilon b^\epsilon) \nabla b^\epsilon}{\epsilon} \right\|_{L_T^1(\dot{B}_{2,1}^\beta)} \\ &\leq C \|b^\epsilon\|_{L_T^{7/(3+3\alpha)}(\dot{B}_{2,1}^{\beta+1})} \|b_{BF}^\epsilon\|_{L_T^{7/(4-3\alpha)}(\dot{B}_{14/(3-4\alpha),1}^{(3-4\alpha)/7})} \\ &\quad + C \|b^\epsilon\|_{L_T^{1/\alpha}(\dot{B}_{2,1}^{\beta+1})} \|b_{HF}^\epsilon\|_{L_T^{1/(1-\alpha)}(\dot{B}_{6/(3-4\alpha),1}^{(3-4\alpha)/3})} \\ &\leq C\epsilon^{\frac{6\alpha-1}{7}} \|b^\epsilon\|_{L_T^{7/(3+3\alpha)}(\tilde{B}_\epsilon^{\beta+1,7/(3+3\alpha)})} \|b_{BF}^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{2+\alpha})} \|b_{BF}^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\alpha-3/4})}^{(4+4\alpha)/7} \\ &\quad + C\epsilon^{2\alpha-1} \|b^\epsilon\|_{L_T^{1/\alpha}(\tilde{B}_\epsilon^{\beta+1,1/\alpha})} \|b_{HF}^\epsilon\|_{L_T^1(\dot{B}_{2,1}^{1+\alpha})}^{1-\frac{4\alpha}{3}} \|b_{HF}^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\alpha-3/4})}^{\frac{4\alpha}{3}} \\ &\leq C\epsilon^\alpha X_\beta \|b^\epsilon\|_{L_T^1(\tilde{B}_\epsilon^{1+\alpha,1})}^{\frac{3-4\alpha}{7}} (\epsilon^{-\frac{1}{4}} \|b^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\alpha-3/4})})^{\frac{4+4\alpha}{7}} \\ &\quad + C\epsilon^\alpha X_\beta \|b^\epsilon\|_{L_T^1(\tilde{B}_\epsilon^{1+\alpha,1})}^{1-\frac{4\alpha}{3}} (\epsilon^{-\frac{1}{4}} \|b_{HF}^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\alpha-3/4})})^{\frac{4\alpha}{3}}. \end{aligned}$$

Finally, we conclude that

$$\begin{aligned} \|T^\epsilon\|_{L_T^1(\dot{B}_{2,1}^\beta)} &\leq C \left(P_0 P_\beta + \epsilon^\alpha (X_\beta + P_\beta) (X_\alpha + \epsilon^{-\frac{1}{4}} Y_\alpha) + \epsilon^\alpha X_\alpha P_0 P_\beta \right. \\ &\quad \left. + \epsilon^{\frac{2}{2+\beta}} (P_0 X_\beta + X_0 P_\beta + P_\beta (X_\alpha + \epsilon^{-\frac{1}{4}} Y_\alpha)) \right). \end{aligned} \quad (3.63)$$

Now we set $p = \frac{1}{\alpha}$ and $r = \frac{2}{2-\alpha}$. Making use of interpolation, the following estimates hold, if $d = 3$,

$$\begin{aligned} \|\nabla u^\epsilon\|_{L_T^{\frac{1}{\alpha}}(\dot{B}_{\infty,1}^{2\alpha-2})} &\leq C \|\nabla \mathcal{P}^\perp u^\epsilon\|_{L_T^{\frac{1}{\alpha}}(\dot{B}_{\infty,1}^{2\alpha-2})} + C \|\nabla \mathcal{P} u^\epsilon\|_{L_T^{\frac{1}{\alpha}}(\dot{B}_{2,1}^{2\alpha-1/2})} \\ &\leq C\epsilon^\alpha (\epsilon^{-\alpha} Y_\alpha^{\frac{1}{\alpha}}) + C P_0; \end{aligned}$$

while if $d = 2$,

$$\begin{aligned} \|\nabla u^\epsilon\|_{L_T^{\frac{1}{\alpha}}(\dot{B}_{\infty,1}^{2\alpha-2})} &\leq C \|\nabla \mathcal{P}^\perp u^\epsilon\|_{L_T^4(\dot{B}_{\infty,1}^{\alpha-\frac{7}{4}})}^{4\alpha} \|\nabla \mathcal{P}^\perp u^\epsilon\|_{L_T^\infty(\dot{B}_{\infty,1}^{\alpha-2})}^{1-4\alpha} + C \|\nabla \mathcal{P} u^\epsilon\|_{L_T^{\frac{1}{\alpha}}(\dot{B}_{2,1}^{2\alpha-1})} \\ &\leq C\epsilon^\alpha (X_\alpha + \epsilon^{-\frac{1}{4}} Y_\alpha) + C P_0. \end{aligned}$$

Meanwhile, for any $d \geq 2$, we have

$$\|\nabla u^\epsilon\|_{L_T^{2/(2-\alpha)}(L^\infty)} \leq C \|u^\epsilon\|_{L_T^{2/(2-\alpha)}(\dot{B}_{2,1}^{d/2+1})} \leq C(P_\alpha + X_\alpha).$$

According to (3.34), we thus have

$$\begin{cases} V_\epsilon^{1/\alpha, 2/(2-\alpha)} \leq C(P_0^{\frac{1}{\alpha}} + \epsilon(\epsilon^{-\alpha} Y_\alpha^{\frac{1}{\alpha}})^{\frac{1}{\alpha}} + (\epsilon^\alpha(P_\alpha + X_\alpha))^{\frac{2}{2-\alpha}}) & \text{if } d = 3, \\ V_\epsilon^{1/\alpha, 2/(2-\alpha)} \leq C(P_0^{\frac{1}{\alpha}} + \epsilon(X_\alpha + \epsilon^{-\frac{1}{4}} Y_\alpha)^{\frac{1}{\alpha}} + (\epsilon^\alpha(P_\alpha + X_\alpha))^{\frac{2}{2-\alpha}}) & \text{if } d = 2. \end{cases}$$

Plugging this latter inequality, (3.41), and (3.62) into (3.33), we eventually find that, for $d = 3$,

$$X_\beta \leq C \exp \left\{ C \left(P_0^\alpha + \epsilon (\epsilon^{-\alpha} Y_\alpha^\alpha)^{\frac{1}{\alpha}} + (\epsilon^\alpha (P_\alpha + X_\alpha))^{\frac{2}{2-\alpha}} \right) \right\} \left(X_\beta^0 + P_0 P_\beta + \epsilon^{\alpha 3} (P_0 X_\beta + (X_\beta + P_\beta)(X_\alpha + \epsilon^{-\alpha} Y_\alpha^{1/\alpha} + \epsilon^{-\frac{2\alpha}{1+2\alpha}} Y_\alpha^{p_\alpha}) + X_\alpha P_0 P_\beta) \right); \quad (3.64)$$

while for $d = 2$, we can apply (3.44) and (3.63) to obtain that

$$X_\beta \leq C \exp \left\{ C \left(P_0^\alpha + \epsilon (X_\alpha + \epsilon^{-\frac{1}{4}} Y_\alpha)^{\frac{1}{\alpha}} + (\epsilon^\alpha (P_\alpha + X_\alpha))^{\frac{2}{2-\alpha}} \right) \right\} \left(X_\beta^0 + P_0 P_\beta + \epsilon^{\alpha 2} (P_0 X_\beta + X_0 P_\beta + (X_\beta + P_\beta)(X_\alpha + \epsilon^{-\frac{1}{4}} Y_\alpha) + X_\alpha P_0 P_\beta) \right), \quad (3.65)$$

where $\alpha_d = \frac{2\alpha}{2+d+2\alpha}$ with $d = 2$ or 3 .

3.4. Bootstrap. The remaining part of the proof works for both dimensions $d = 3$ and $d = 2$. Set

$$X := X_0 + X_\alpha, \quad V := V_0 + V_\alpha, \quad W := W_0 + W_\alpha, \quad X^0 := X_0^0 + X_\alpha^0.$$

With these new notations, by combining together (3.5) or (3.6), (3.30) or (3.31), and (3.64) or (3.65), we conclude that

$$W \leq C e^{C(V+X)} \left(W^2 (1 + W + W^2 + W^4 + \epsilon^{\alpha_d} (V + V^2 + V^4)) + \epsilon^{\alpha_d} (X^2 + (X^0)^2 + X^4 + V(X + X^2 + X^0 + XV^2 + V^2 + V^3)) \right), \quad (3.66)$$

$$\begin{aligned} X &\leq C \exp \left\{ C \left(\epsilon (X + X^2)^{\frac{1}{\alpha}} + \epsilon (X(V + W)^2)^{\frac{1}{\alpha}} + (\epsilon^\alpha X)^{\frac{2}{2-\alpha}} \right) \right\} \\ &\quad \times \exp \left\{ C \left((V + W)^{\frac{1}{\alpha}} + \epsilon (X^0 + (V + W)^2)^{\frac{1}{\alpha}} + \epsilon^{\frac{2\alpha}{2-\alpha}} (V + W)^{\frac{2}{2-\alpha}} \right) \right\} \\ &\quad \times \left(X^0 + (V + W)(V + W + \epsilon^{\alpha_d} (X^0 + V^2 + W^2)) + \epsilon^{\alpha_d} X (X^0 + X + X^2 + (1 + X)(W + V + W^2 + V^2) + V^3 + W^3) \right). \end{aligned} \quad (3.67)$$

In order to get a bound for $(b^\epsilon, u^\epsilon, H^\epsilon)$, we need a bootstrap argument. More precisely, we have the following lemma.

Lemma 3.1. *Suppose that $(v, B) \in F_{T_0}^{\frac{d}{2}} \cap F_{T_0}^{\frac{d}{2}+\alpha}$ for some finite or infinite T_0 . Then, there exists an $\epsilon_0 > 0$, depending only on $\alpha, d, V(T_0)$, and the norm of $(b_0, \mathcal{P}^\perp u_0, H_0)$ in*

$$\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha} \times (\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})^d \times (\dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}-1+\alpha})^d$$

such that if $\epsilon \leq \epsilon_0$, $(b^\epsilon, u^\epsilon, H^\epsilon) \in E_{\epsilon, T}^{\frac{d}{2}} \cap E_{\epsilon, T}^{\frac{d}{2}+\alpha}$, and $\epsilon |b^\epsilon| \leq 3/4$ for some $T \leq T_0$, the following estimates hold with the constant $C = C(d, \mu, \lambda, \nu, P, \alpha)$ appearing in (3.66) and (3.67):

$$\begin{aligned} X_T &\leq X_M := 16C e^{C V^{\frac{1}{\alpha}}(T_0)} (X^0 + V^2(T_0)), \\ \epsilon^{-\alpha_d} W_T &\leq W_M := 4C e^{C(V(T_0)+X_M)} \left(X_M^2 + X_0^2 + X_M^4 + V(T_0)(X_M + X_M^2 + X_0 + X_M V^2(T_0) + V^2(T_0) + V^3(T_0)) \right). \end{aligned}$$

Proof. Let $I := \{t \leq T | X(t) \leq X_M \text{ and } W(t) \leq \epsilon^{\alpha_d} W_M\}$. Obviously, X and W are continuous nondecreasing functions so that if, say, $C \geq 1$, then I is a closed interval of \mathbb{R}^+ with lower bound 0.

Let $T^* := \sup I$. Choose ϵ sufficiently small so that the following conditions are satisfied:

$$\begin{aligned}
& C e^{C(V(T_0)+X_M)} \epsilon^{\alpha_d} W_M \left(1 + W_M + W_M^2 + W_M^4 \right. \\
& \quad \left. + \epsilon^{\alpha_d} (V(T_0) + V^2(T_0) + V^4(T_0)) \right) \leq \frac{1}{2}, \\
& \exp \left\{ C(\epsilon(X_M + X_M^2)^{1/\alpha} + \epsilon(X_M(V(T_0) + \epsilon^{\alpha_d} W_M)^2)^{1/\alpha} + (\epsilon^\alpha X_M)^{2/(2-\alpha)}) \right\} \leq 2, \\
& \exp \left\{ C((V(T_0) + \epsilon^{\alpha_d} W_M)^{1/\alpha} + \epsilon(X^0 + (V(T_0) + \epsilon^{\alpha_d} W_M)^2)^{1/\alpha} \right. \\
& \quad \left. + \epsilon^{2\alpha/(2-\alpha)} (V(T_0) + \epsilon^{\alpha_d} W_M)^{2/(2-\alpha)}) \right\} \leq 2e^{CV^{\frac{1}{\alpha}}(T_0)}, \\
& X_0 + (V(T_0) + \epsilon^{\alpha_d} W_M)(V(T_0) + \epsilon^{\alpha_d} W_M + \epsilon^{\alpha_d} (X^0 + V^2(T_0) + \epsilon^{2\alpha_d} W_M^2)) \\
& \quad \leq 2(X^0 + V^2(T_0)), \\
& C e^{CV^{1/\alpha}(T_0)} \epsilon^{\alpha_d} \left\{ X^0 + X_M + X_M^2 + (1 + X_M) \right. \\
& \quad \left. \times (V(T_0) + V^2(T_0) + \epsilon^{\alpha_d} W_M + \epsilon^{2\alpha_d} W_M^2) + V^3(T_0) + \epsilon^{3\alpha_d} W_M^3 \right\} \leq \frac{1}{12}.
\end{aligned}$$

Then, by the (3.66) and (3.67), we obtain that

$$\begin{aligned}
X(T^*) & \leq 12C e^{CV^{1/\alpha}(T_0)} (X^0 + V^2(T)), \\
W(T^*) & \leq 2C e^{C(V(T_0)+X_M)} \epsilon^{\alpha_d} (X_M^2 + X_0^2 + X_M^4 \\
& \quad + V(T_0)(X_M + X_M^2 + X_0 + X_M V^2(T_0) + V^2(T_0) + V^3(T_0))).
\end{aligned}$$

In other words, at time T^* the desired inequalities are strict. Hence, we must have $T^* = T$. \square

3.5. Continuation argument. First, we have to establish the existence of a local solution in $E_{\epsilon,T}^{\frac{d}{2}} \cap E_{\epsilon,T}^{\frac{d}{2}+\alpha}$. Making the change of function $a^\epsilon = \epsilon b^\epsilon$, Theorem 1.1 will enable us to get a local solution $(b^\epsilon, u^\epsilon, H^\epsilon)$ on $[0, T] \times \mathbb{R}^d$ which belongs to E_T^α and satisfies

$$1 + \epsilon \inf_{(t,x) \in [0,T] \times \mathbb{R}^d} b^\epsilon(t, x) > 0.$$

Moreover, due to the facts that $b_0 \in \dot{B}_{2,1}^{\frac{d}{2}-1}$ and $\partial_t b^\epsilon + u^\epsilon \cdot \nabla b^\epsilon \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})$, we readily get that $b^\epsilon \in \mathcal{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1})$. Therefore, $(b^\epsilon, u^\epsilon, H^\epsilon) \in E_{\epsilon,T}^{\frac{d}{2}} \cap E_{\epsilon,T}^{\frac{d}{2}+\alpha}$. Now, assuming that we have $(v, B) \in F_{T_0}^{\frac{d}{2}} \cap F_{T_0}^{\frac{d}{2}+\alpha}$ for some $T_0 \in (0, +\infty]$, we shall prove that the lifespan T_ϵ satisfies $T_\epsilon \geq T_0$ if ϵ is sufficiently small, where T_ϵ is the supremum of the set

$$\left\{ T \in \mathbb{R}^+ \mid (b^\epsilon, u^\epsilon, H^\epsilon) \in E_{\epsilon,T}^{\frac{d}{2}} \cap E_{\epsilon,T}^{\frac{d}{2}+\alpha} \text{ and } \forall (t, x) \in [0, T] \times \mathbb{R}^d, |\epsilon b^\epsilon| \leq \frac{3}{4} \right\}.$$

We suppose that T_ϵ is finite and satisfies $T_\epsilon \leq T_0$. Thanks to Lemma 3.1, we have, for any $T < T_\epsilon$ and $\epsilon \leq \epsilon_0$, that

$$X(T) \leq X_M \quad \text{and} \quad W(T) \leq \epsilon^{\alpha_d} W_M.$$

From the first inequality and (3.12), we conclude that

$$\epsilon \|b^\epsilon\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \leq \epsilon^\alpha X_M.$$

Obviously, we require that, for ϵ_0 sufficiently small,

$$1 + \epsilon \inf_{(t,x) \in [0, T_\epsilon] \times \mathbb{R}^d} |b^\epsilon(t, x)| > 0.$$

Since $\epsilon b^\epsilon \in L_{T_\epsilon}^\infty(\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha})$, $\nabla u^\epsilon \in L_{T_\epsilon}^1(\dot{B}_{2,1}^{\frac{d}{2}})$, and $\nabla u^\epsilon \in L_{T_\epsilon}^1(\dot{B}_{2,1}^{\frac{d}{2}})$, the continuation criterion stated in Proposition 2.6 ensures that $(b^\epsilon, u^\epsilon, H^\epsilon)$ may be continued beyond T_ϵ , which contradicts definition of T_ϵ . Therefore, $T_\epsilon \geq T_0$ for $\epsilon \leq \epsilon_0$.

The proof of Theorem 1.2 is now completed. \square

APPENDIX A. BASIC FACTS ON BESOV SPACES

In this section we recall the definition and some basic properties of homogeneous Besov space. Most of the materials stated below can be found in the books [1, 4, 36]. We collect them below for the reader's convenience.

Definition A.1 ([36]). Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity that satisfies $\hat{\phi} \in C_0^\infty(B_2 \setminus B_{1/2})$, $\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi)$ and $\sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1$ for any $\xi \neq 0$, where $\hat{\phi}$ is the Fourier transform of ϕ and B_r is the ball with radius r centered at the origin. The homogeneous Besov space is defined as

$$\dot{B}_{p,q}^s := \left\{ f \in \mathcal{S}'/\mathbf{P} : \|f\|_{\dot{B}_{p,q}^s} < \infty \right\}$$

with the norm

$$\|f\|_{\dot{B}_{p,q}^s} := \begin{cases} \left\{ \sum_{j \in \mathbb{Z}} \|2^{js} \phi_j * f\|_{L^p}^q \right\}^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \sup_{j \in \mathbb{Z}} \|2^{js} \phi_j * f\|_{L^\infty}, & q = \infty, \end{cases}$$

for all $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, where \mathcal{S}' is the space of tempered distributions and \mathbf{P} is the space of polynomials.

Definition A.2 ([36]). For $T > 0$, $s \in \mathbb{R}$, and $1 \leq r, \rho \leq \infty$, we set

$$\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} := \|2^{js} \|\dot{\Delta}_j u\|_{L_T^\rho(L^p)}\|_{l^r(\mathbb{Z})}.$$

We can then define the space $\tilde{L}_T^\rho(\dot{B}_{p,r}^s)$ as the set of tempered distributions u over $(0, T) \times \mathbb{R}^d$ such that $\lim_{j \rightarrow -\infty} \dot{S}_j u = 0$ in $L_T^\rho(L^\infty)$ and $\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} < +\infty$.

Remark A.1 ([36]). The spaces $\tilde{L}_T^\rho(\dot{B}_{p,r}^s)$ may be linked with the more classical spaces $L_T^\rho(\dot{B}_{p,r}^s)$ via the Minkowski inequality:

$$\|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \leq \|u\|_{L_T^\rho(\dot{B}_{p,r}^s)} \quad \text{if } r \geq \rho, \quad \|u\|_{\tilde{L}_T^\rho(\dot{B}_{p,r}^s)} \geq \|u\|_{L_T^\rho(\dot{B}_{p,r}^s)} \quad \text{if } r \leq \rho.$$

The general principles is that all the properties of continuity for the product, composition, remainder, and paraproduct remain true in these spaces. The exponent ρ just has to behave according to Hölder's inequality for the time variable.

Proposition A.1 ([10]). *The following properties hold:*

(1) *Derivation: there exists a universal constant C such that*

$$C^{-1}\|u\|_{\dot{B}_{p,1}^s} \leq \|\nabla u\|_{\dot{B}_{p,1}^{s-1}} \leq C\|u\|_{\dot{B}_{p,1}^s};$$

(2) *Fractional derivation: let $\Lambda := \sqrt{-\Delta}$ and $\sigma \in \mathbb{R}$, then the operator Λ^σ is an isomorphism from $\dot{B}_{p,1}^s$ to $\dot{B}_{p,1}^{s-\sigma}$;*

(3) *Sobolev embeddings: if $p_1 < p_2$ then $\dot{B}_{p,1}^s \hookrightarrow \dot{B}_{p,2}^{s-d(1/p_1-1/p_2)}$;*

(4) *Interpolation : $[\dot{B}_{p,1}^{s_1}, \dot{B}_{p,1}^{s_2}]_\theta = \dot{B}_{p,1}^{\theta s_1 + (1-\theta)s_2}$;*

(5) *Algebraic properties: for $s > 0$, $\dot{B}_{p,1}^s \cap L^\infty$ is an algebra;*

(6) *Scaling properties:*

(a) *for all $\lambda > 0$ and $u \in \dot{B}_{p,1}^s$, we have*

$$\|u(\lambda \cdot)\|_{\dot{B}_{p,1}^s} \approx (\lambda)^{s-d/p} \|u\|_{\dot{B}_{p,1}^s},$$

(b) *for $u = u(t, x)$ in $L_T^r(\dot{B}_{p,1}^s)$, we have*

$$\|u(\lambda^a \cdot, \lambda^b \cdot)\|_{L_T^r(\dot{B}_{p,1}^s)} \approx \lambda^{b(s-d/p)-a/r} \|u\|_{L_{\lambda^a T}^r(\dot{B}_{p,1}^s)}.$$

Let us state some continuity results for the product.

Proposition A.2 ([10]). *If $u \in \dot{B}_{p_1,1}^{s_1}$ and $v \in \dot{B}_{p_2,1}^{s_2}$ with $1 \leq p_1 \leq p_2 \leq +\infty, s_1 \leq d/p_1, s_2 \leq d/p_2$ and $s_1 + s_2 > 0$. then $uv \in \dot{B}_{p_2,1}^{s_1+s_2-d/p_1}$ and*

$$\|uv\|_{\dot{B}_{p_2,1}^{s_1+s_2-d/p_1}} \leq C\|u\|_{\dot{B}_{p_1,1}^{s_1}} \|v\|_{\dot{B}_{p_2,1}^{s_2}}.$$

If $u \in \dot{B}_{p_1,1}^{s_1} \cap \dot{B}_{p_2,1}^{s_2}$ and $v \in \dot{B}_{p_2,1}^{t_1} \cap \dot{B}_{p_2,1}^{t_2}$ with $1 \leq p_1, p_2 \leq +\infty, s_1, t_1 \leq d/p_1$ and $s_1 + t_2 = s_2 + t_1 > d \max\{0, \frac{1}{p_1} + \frac{1}{p_2} - 1\}$, then $uv \in \dot{B}_{p_2,1}^{s_1+t_2-d/p_1}$ and

$$\|uv\|_{\dot{B}_{p_2,1}^{s_1+t_2-d/p_1}} \leq C\|u\|_{\dot{B}_{p_1,1}^{s_1}} \|v\|_{\dot{B}_{p_2,1}^{t_2}} + \|u\|_{\dot{B}_{p_2,1}^{s_2}} \|v\|_{\dot{B}_{p_1,1}^{t_1}}.$$

Moreover, if $s_1 = 0$ and $p_1 = +\infty$, then $\|u\|_{\dot{B}_{\infty,1}^0}$ may be replaced with $\|u\|_{L^\infty}$.

Definition A.3 ([36]). Let $s \in \mathbb{R}, \alpha > 0$ and $1 \leq r \leq +\infty$ and

$$\|u\|_{\tilde{B}_\alpha^{s,r}} := \sum_{q \in \mathbb{Z}} 2^{qs} \max\{\alpha, 2^{-q}\}^{1-2/r} \|\dot{\Delta}_q u\|_{L^2}.$$

Let $m = -[d/2 + 2 - 2/r - s]$, we then define

$$\begin{aligned} \tilde{B}_\alpha^{s,r}(\mathbb{R}^d) &:= \{u \in \mathcal{S}'(\mathbb{R}^d) \mid \|u\|_{\tilde{B}_\alpha^{s,r}} < +\infty\} & \text{if } m < 0, \\ \tilde{B}_\alpha^{s,r}(\mathbb{R}^d) &:= \{u \in \mathcal{S}'(\mathbb{R}^d)/\mathbf{P}_m(\mathbb{R}^d) \mid \|u\|_{\tilde{B}_\alpha^{s,r}} < +\infty\} & \text{if } m \geq 0. \end{aligned}$$

We will use the following high-low frequencies decomposition:

$$u_{BF} := \sum_{q \leq [-\log_2 \alpha]} \dot{\Delta}_q u, \quad u_{HF} := \sum_{q > [-\log_2 \alpha]} \dot{\Delta}_q u.$$

Remark A.2 ([10]). (i) $\tilde{B}_\alpha^{s,2} = \dot{B}_{2,1}^s$;

(ii) If $r \geq 2$ then $\tilde{B}_\alpha^{s,r} = \dot{B}_{2,1}^{s+2/r-1} \cap \dot{B}_{2,1}^s$ and

$$\|u\|_{\tilde{B}_\alpha^{s,r}} \approx \|u\|_{\dot{B}_{2,1}^{s+2/r-1}} + \alpha^{1-2/r} \|u\|_{\dot{B}_{2,1}^s},$$

If $r \leq 2$ then $\tilde{B}_\alpha^{s,r} = \dot{B}_{2,1}^{s+2/r-1} + \dot{B}_{2,1}^s$ and

$$\|u\|_{\tilde{B}_\alpha^{s,r}} \approx \|u_{BF}\|_{\dot{B}_{2,1}^{s+2/r-1}} + \alpha^{1-2/r} \|u_{HF}\|_{\dot{B}_{2,1}^s};$$

(iii) For all $\lambda > 0$ and $u \in \tilde{B}_\alpha^{s,r}$, we have

$$\|u(\lambda \cdot)\|_{\tilde{B}_\alpha^{s,r}} \approx \lambda^{s-d/2+2/r-1} \|u\|_{\tilde{B}_{\lambda\alpha}^{s,r}}.$$

The paraproduct between u and v is given by

$$\dot{T}_u v := \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} u \dot{\Delta}_q v, \quad \dot{R}(u, v) := \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u \tilde{\Delta}_q v,$$

with $\tilde{\Delta}_q = \dot{\Delta}_{q-1} + \dot{\Delta}_q + \dot{\Delta}_{q+1}$. We have the following Bony decomposition (modulo a polynomial):

$$uv = \dot{T}_u v + \dot{T}_u v + \dot{R}(u, v).$$

The notation $\dot{T}'_u v := \dot{T}_u v + \dot{R}(u, v)$ will be employed likewise.

Remark A.3 ([10]). Let $1 \leq p_1, p_2 \leq +\infty$. For all $s_2 \in \mathbb{R}$ and $s_1 \leq d/p_1$, we have

$$\|\dot{T}_u v\|_{\dot{B}_{p_2,1}^{s_1+s_2}} \leq C \|u\|_{\dot{B}_{\infty,\infty}^{s_1}} \|v\|_{\dot{B}_{p_2,1}^{s_2}}.$$

If $(s_1, s_2) \in \mathbb{R}^2$ satisfies $s_1 + s_2 > d \max\{0, \frac{1}{p_1} + \frac{1}{p_2} - 1\}$, then

$$\|\dot{R}(u, v)\|_{\dot{B}_{p_2,1}^{s_1+s_2-d/p_1}} \leq C \|u\|_{\dot{B}_{p_1,\infty}^{s_1}} \|v\|_{\dot{B}_{p_2,1}^{s_2}}.$$

Proposition A.3 ([1]). Let K be a compact subset of \mathbb{R}^d . Denote by $B_{p,r}^s(K)$ [resp., $\dot{B}_{p,r}^s(K)$] the set of distributions u in $B_{p,r}^s$ (resp., $\dot{B}_{p,r}^s$), the support of which is included in K . If $s > 0$, then the spaces $B_{p,r}^s(K)$ and $\dot{B}_{p,r}^s(K)$ coincide. Moreover, a constant C exists such that for any u in $\dot{B}_{p,r}^s(K)$,

$$\|u\|_{B_{p,r}^s} \leq C(1 + |K|)^{\frac{s}{d}} \|u\|_{\dot{B}_{p,r}^s}.$$

Here $B_{p,r}^s$ denotes the inhomogeneous Besov space.

Proposition A.4 ([1]). If $s' < s$, then for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, multiplication by φ is a compact operator from $\dot{B}_{p,\infty}^s$ to $\dot{B}_{p,1}^{s'}$.

Proposition A.5 (Fatou's property [1]). Let $(s_1, s_2) \in \mathbb{R}^2$ and $1 \leq p_1, p_2, r_1, r_2 \leq \infty$. Assume that (s_1, p_1, r_1) satisfies $s_1 < \frac{d}{p_1}$, or $s_1 = \frac{d}{p_1}$ and $r_1 = 1$. the space $\dot{B}_{p_1,r_1}^{s_1} \cap \dot{B}_{p_2,r_2}^{s_2}$ endowed with the norm $\|\cdot\|_{\dot{B}_{p_1,r_1}^{s_1}} + \|\cdot\|_{\dot{B}_{p_2,r_2}^{s_2}}$ is then complete and satisfies the Fatou's property: If $(u_n)_{n \geq 1}$ is bounded sequence of $\dot{B}_{p_1,r_1}^{s_1} \cap \dot{B}_{p_2,r_2}^{s_2}$, then an element u of $\dot{B}_{p_1,r_1}^{s_1} \cap \dot{B}_{p_2,r_2}^{s_2}$ and a subsequence $u_{\psi(n)}$ exist such that $\lim_{n \rightarrow \infty} u_{\psi(n)} = u$ in \mathcal{S}' and $\|u\|_{\dot{B}_{p_k,r_k}^{s_k}} \leq C \liminf_{n \rightarrow \infty} \|u_{\psi(n)}\|_{\dot{B}_{p_k,r_k}^{s_k}}$ for $k = 1, 2$.

Lemma A.1 (Bernstein inequality [1]). *Let \mathcal{C} be an annulus and B a ball, A constant C exists such that for any nonnegative integer k , any couple (p, q) in $[1, \infty]^2$ with $q \geq p \geq 1$, and any function u of L^p , we have*

$$\text{Supp } \hat{u} \subset \lambda B \Rightarrow \|D^k u\|_{L^q} := \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p},$$

$$\text{Supp } \hat{u} \subset \lambda \mathcal{C} \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$

We also need the following composition propositions in $\dot{B}_{p,1}^s$.

Lemma A.2 ([10]). *Let $s > 0$, $p \in [1, +\infty]$, $u \in \dot{B}_{p,1}^s \cap L^\infty$, and $F \in W_{loc}^{[s]+2,\infty}$ such that $F(0) = 0$. Then $F(u) \in \dot{B}_{p,1}^s$ and there exists a constant $C = C(s, p, d, F, \|u\|_{L^\infty})$ such that*

$$\|F(u)\|_{\dot{B}_{p,1}^s} \leq C \|u\|_{\dot{B}_{p,1}^s}.$$

Lemma A.3 ([10]). *Let f be a smooth function such that $f'(0) = 0$, s be a positive real number and (p, r) in $[1, \infty]^2$ such that $s < \frac{d}{p}$, or $s = \frac{d}{p}$ and $r = 1$. For any couple (u, v) of functions in $\dot{B}_{p,r}^s \cap L^\infty$, the function $f \circ v - f \circ u$ then belongs to $\dot{B}_{p,r}^s \cap L^\infty$ and*

$$\begin{aligned} \|f(u) - f(v)\|_{\dot{B}_{p,r}^s} &\leq C (\|v - u\|_{\dot{B}_{p,r}^s} \sup_{\tau \in [0,1]} \|u + \tau(v - u)\|_{L^\infty} \\ &\quad + \|v - u\|_{L^\infty} \sup_{\tau \in [0,1]} \|u + \tau(v - u)\|_{\dot{B}_{p,r}^s}), \end{aligned}$$

where C depends on f'' , $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$.

Lemma A.4 ([10]). *Let \mathcal{C}' be an annulus and $(u_j)_{j \in \mathbb{Z}}$ be a sequence of functions such that*

$$\text{Supp } \hat{u}_j \subset 2^j \mathcal{C}' \quad \text{and} \quad \|(2^{js} \|u_j\|_{L^p})_{j \in \mathbb{Z}}\|_{l^r} < \infty.$$

If the series $\sum_{j \in \mathbb{Z}} u_j$ converges in \mathcal{S}' to some u in \mathcal{S}'_h , then u is in $\dot{B}_{p,r}^s$ and

$$\|u\|_{\dot{B}_{p,r}^s} \leq C \|(2^{js} \|u_j\|_{L^p})_{j \in \mathbb{Z}}\|_{l^r}.$$

APPENDIX B. LOCAL EXISTENCE RESULTS FOR INCOMPRESSIBLE MHD EQUATIONS

Proposition B.1. *Let $\alpha \geq 0$, $d = 2$ or 3 . and $(u_0, b_0) \in \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+\alpha-1}$ be two divergence-free vector fields. Then there exists a time T and a unique local solution (u, b) to the following initial value problem*

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u - b \cdot \nabla b - \nabla P = 0, \\ \partial_t b - \Delta b + u \cdot \nabla b - b \cdot \nabla u = 0, \\ \text{div } u = 0, \quad \text{div } b = 0, \\ (u, b)|_{t=0} = (u_0, b_0), \quad x \in \mathbb{R}^d, \end{cases} \quad (\text{B.1})$$

such that $(u, b) \in F_T^{\frac{d}{2}} \cap F_T^{\frac{d}{2}+\alpha}$ and there exist two constants c and C depending only on d such that the time T is bounded from below by

$$\sup \left\{ T' > 0 \left| \sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{2}-1)} (1 - e^{-c2^{2j}T'})^{\frac{1}{2}} (\|\dot{\Delta}_j u_0\|_{L^2} + \|\dot{\Delta}_j b_0\|_{L^2}) \right\} \leq C.$$

Proof. We shall adopt the fixed point method. Denote by $e^{t\Delta}$ the semi-group of the heat equation. Let $(u_L, b_L) \in F_T^{\frac{d}{2}} \cap F_T^{\frac{d}{2}+\alpha}$ be the solution of

$$\begin{cases} \partial_t u_L - \Delta u_L = 0, \\ \partial_t b_L - \Delta b_L = 0, \\ (u_L, b_L)|_{t=0} = (u_0, b_0)(x), \quad x \in \mathbb{R}^d. \end{cases}$$

Assume that the time $T \in (0, +\infty]$ has been chosen in such a way that

$$\|(u_L, b_L)\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \leq \frac{1}{4C} \quad (\text{B.2})$$

for a constant C to be defined below.

Let $0 < R < \frac{1}{4C}$ and $R_\alpha := \|(u_L, b_L)\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\alpha})}$. Let \mathcal{G} be the set of divergence-free vector fields with coefficients in $F_T^{\frac{d}{2}} \cap F_T^{\frac{d}{2}+\alpha}$, and such that $\|(u, b)\|_{F_T^{\frac{d}{2}}} \leq R$ and $\|(u, b)\|_{F_T^{\frac{d}{2}+\alpha}} \leq R_\alpha$. Define

$$\mathcal{F}(\bar{u}, \bar{b}) := \left(\int_0^t e^{(t-\tau)\Delta} \mathcal{P}(b \cdot \nabla b - u \cdot \nabla u) d\tau, \int_0^t e^{(t-\tau)\Delta} \mathcal{P}(b \cdot \nabla u - u \cdot \nabla b) d\tau \right)$$

with $u = \bar{u} + u_L$ and $b = \bar{b} + b_L$. According to Propositions A.2 and A.3, \mathcal{F} maps $F_T^{\frac{d}{2}} \cap F_T^{\frac{d}{2}+\alpha}$ into itself, and, for $\beta = 0$ or α , we have

$$\begin{aligned} \|\mathcal{F}(\bar{u}, \bar{b})\|_{F_T^{\frac{d}{2}+\beta}} &\leq C \left(\|(\bar{u}, \bar{b})\|_{F_T^{\frac{d}{2}}} + \|(u_L, b_L)\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \right) \\ &\quad \times \left(\|(\bar{u}, \bar{b})\|_{F_T^{\frac{d}{2}+\beta}} + \|(u_L, b_L)\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}+\beta})} \right). \end{aligned}$$

Hence it is easy to check that \mathcal{F} maps \mathcal{G} to \mathcal{G} . Similar computations imply that

$$\begin{aligned} \|\mathcal{F}(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}}} &\leq C \left(\|(\bar{u}_1, \bar{b}_1)\|_{F_T^{\frac{d}{2}}} + \|(\bar{u}_2, \bar{b}_2)\|_{F_T^{\frac{d}{2}}} + \|(\bar{u}_L, \bar{b}_L)\|_{L_T^2(\dot{B}_T^{\frac{d}{2}})} \right) \\ &\quad \times \|(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}}}, \\ \|\mathcal{F}(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}+\alpha}} &\leq C \left(\|(\bar{u}_2, \bar{b}_2)\|_{F_T^{\frac{d}{2}+\alpha}} + \|(\bar{u}_L, \bar{b}_L)\|_{L_T^2(\dot{B}_T^{\frac{d}{2}+\alpha})} \right) \\ &\quad \times \|(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}}} \\ &\quad + C \left(\|(\bar{u}_1, \bar{b}_1)\|_{F_T^{\frac{d}{2}}} + \|(\bar{u}_L, \bar{b}_L)\|_{L_T^2(\dot{B}_T^{\frac{d}{2}})} \right) \\ &\quad \times \|(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}+\alpha}}. \end{aligned}$$

Set $k = \frac{1}{2} + 2RC$ and $K = 4R_\alpha C$. According to the above inequalities, we have, for all $\eta > 0$,

$$\begin{aligned} &\|\mathcal{F}(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}}} + \eta \|\mathcal{F}(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}+\alpha}} \\ &\leq C \|(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}}} \left(\sum_{i=1}^2 \|(\bar{u}_i, \bar{b}_i)\|_{F_T^{\frac{d}{2}}} + \|(\bar{u}_L, \bar{b}_L)\|_{L_T^2(\dot{B}_T^{\frac{d}{2}})} \right) \end{aligned}$$

$$\begin{aligned}
& + \eta \|(\bar{u}_2, \bar{b}_2)\|_{F_T^{\frac{d}{2}+\alpha}} + \eta \|(\bar{u}_L, \bar{b}_L)\|_{L_T^2(\dot{B}_T^{\frac{d}{2}+\alpha})} \\
& + C\eta \|(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}+\alpha}} (\|(\bar{u}_1, \bar{b}_1)\|_{F_T^{\frac{d}{2}}} + \|(\bar{u}_L, \bar{b}_L)\|_{L_T^2(\dot{B}_T^{\frac{d}{2}})}) \\
& \leq C (\|(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}}} + \eta \|(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}+\alpha}}) \\
& \quad \times \left(\sum_{i=1}^2 \|(\bar{u}_i, \bar{b}_i)\|_{F_T^{\frac{d}{2}}} + \|(\bar{u}_L, \bar{b}_L)\|_{L_T^2(\dot{B}_T^{\frac{d}{2}})} \right. \\
& \quad \left. + \eta \|(\bar{u}_2, \bar{b}_2)\|_{F_T^{\frac{d}{2}+\alpha}} + \eta \|(\bar{u}_L, \bar{b}_L)\|_{L_T^2(\dot{B}_T^{\frac{d}{2}+\alpha})} \right) \\
& \leq (k + \eta K) \left(\|(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}}} + \eta \|(\bar{u}_1 - \bar{u}_2, \bar{b}_1 - \bar{b}_2)\|_{F_T^{\frac{d}{2}+\alpha}} \right).
\end{aligned}$$

Choosing η and R sufficiently small such that $k + \eta K < 1$, we conclude that \mathcal{F} is a contraction map on \mathcal{G} endowed with the norm $\|\cdot\|_{F_T^{\frac{d}{2}}} + \eta \|\cdot\|_{F_T^{\frac{d}{2}+\alpha}}$. Denoting

$$u = \bar{u} + u_L, \quad b = \bar{b} + b_L,$$

where (\bar{u}, \bar{b}) is the unique point of \mathcal{F} in \mathcal{G} , we easily find that (u, b) solves the problem (B.1).

Now, according to Proposition 2.3 in [7], we have, for the two constants c and C depending only on d ,

$$\begin{aligned}
\|u_L\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} & \leq C \left(\sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{2}-1)} (1 - e^{-c2^{2j}T})^{\frac{1}{2}} \right) \|\dot{\Delta}_j u_0\|_{L^2}, \\
\|b_L\|_{L_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} & \leq C \left(\sum_{j \in \mathbb{Z}} 2^{j(\frac{d}{2}-1)} (1 - e^{-c2^{2j}T})^{\frac{1}{2}} \right) \|\dot{\Delta}_j b_0\|_{L^2}.
\end{aligned}$$

Thanks to the Lebesgue's dominated convergence theorem, the right-hand sides on the above equalities tend to zero as T tends to zero. Combining this with (B.2) gives us a bound from below for the life span of (u, b) . The uniqueness of solution can be proved in a standard way. \square

Below we give a priori estimates for the following initial value problem:

$$\begin{cases} \partial_t w - \mu \Delta w + A \cdot \nabla w + w \cdot \nabla A - B \cdot \nabla E - E \cdot \nabla B = f, \\ \partial_t B - \nu \Delta B + A \cdot \nabla B - B \cdot \nabla A + w \cdot \nabla E - E \cdot \nabla w = g, \\ (w, B)|_{t=0} = (w_0, B_0)(x), \quad x \in \mathbb{R}^d. \end{cases} \quad (\text{B.3})$$

Proposition B.2. *Let $s \in (-\frac{d}{2}, \frac{d}{2}]$. Assume that $w_0, B_0 \in \dot{B}_{2,1}^s$, $f, g \in L_T^1(\dot{B}_{2,1}^s)$, and $A, E \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})$ are time-dependent vector fields. Then there exists a universal constant κ , and a constant C depending only on d and s , such that, for all $t \in [0, T]$,*

$$\begin{aligned}
\|(w, B)\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^s)} + \kappa \mathcal{L} \|(w, B)\|_{L_t^1(\dot{B}_{2,1}^{s+2})} & \leq \left(\|(w_0, B_0)\|_{\dot{B}_{2,1}^s} + \|(f, g)\|_{L_t^1(\dot{B}_{2,1}^s)} \right) \\
& \quad \times \exp \left(C \int_0^t (\|\nabla A\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla E\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) d\tau \right)
\end{aligned}$$

with $\underline{\nu} := \min\{\mu, \nu\}$.

Proof. The desired estimate will be obtained after localizing the equations (B.3) by means of the homogeneous Littlewood-Paley decomposition. More precisely, applying $\dot{\Delta}_j$ to (B.3) yields

$$\begin{cases} \partial_t w_j - \mu \Delta w_j + A \cdot \nabla w_j - E \cdot \nabla B_j \\ \quad = f_j - \dot{\Delta}_j(w \cdot \nabla A) + \dot{\Delta}_j(B \cdot \nabla E) + R_j^1 - R_j^2, \\ \partial_t B_j - \nu \Delta B_j + A \cdot \nabla B_j - E \cdot \nabla w_j \\ \quad = g_j + \dot{\Delta}_j(B \cdot \nabla A) - \dot{\Delta}_j(w \cdot \nabla E) + R_j^3 - R_j^4, \end{cases}$$

with

$$\begin{aligned} w_j &:= \dot{\Delta}_j w, & B_j &:= \dot{\Delta}_j B, \\ R_j^1 &:= \sum_k [A^k, \dot{\Delta}_j] \partial_k w, & R_j^2 &:= \sum_k [E^k, \dot{\Delta}_j] \partial_k B, \\ R_j^3 &:= \sum_k [A^k, \dot{\Delta}_j] \partial_k B, & R_j^4 &:= \sum_k [E^k, \dot{\Delta}_j] \partial_k w. \end{aligned}$$

Taking the L^2 inner product of the above equations with w_j and B_j , respectively, we easily get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|w_j\|_{L^2}^2 + \|B_j\|_{L^2}^2) + \mu \int |\nabla w_j|^2 dx + \nu \int |\nabla B_j|^2 dx \\ &= \frac{1}{2} \int (\operatorname{div} A)(|w_j|^2 + |B_j|^2) dx + \int f_j w_j dx + \int g_j B_j dx - \int \operatorname{div} E (B \cdot w) dx \\ & \quad - \int \dot{\Delta}_j(w \cdot \nabla A) w_j dx + \int \dot{\Delta}_j(B \cdot \nabla E) w_j dx - \int \dot{\Delta}_j(w \cdot \nabla E) B_j dx \\ & \quad + \int \dot{\Delta}_j(B \cdot \nabla A) B_j dx + \int (R_j^1 - R_j^2) w_j dx + \int (R_j^3 - R_j^4) w_j dx. \end{aligned}$$

Hence, thanks to the Bernstein's inequality, we get, for some universal constant κ ,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|w_j\|_{L^2}^2 + \|B_j\|_{L^2}^2) + \kappa \underline{\nu} 2^{2j} (\|w_j\|_{L^2}^2 + \|B_j\|_{L^2}^2) \\ & \leq (\|f_j\|_{L^2} + \|\operatorname{div} A\|_{L^\infty} \|w_j\|_{L^2} \\ & \quad + \|\dot{\Delta}_j(w \cdot \nabla A)\|_{L^2} + \|\dot{\Delta}_j(B \cdot \nabla E)\|_{L^2} + \|R_j^1\|_{L^2} + \|R_j^2\|_{L^2}) \|w_j\|_{L^2}^2 \\ & \quad + (\|g_j\|_{L^2} + \|\operatorname{div} A\|_{L^\infty} \|B_j\|_{L^2} + \|\dot{\Delta}_j(w \cdot \nabla E)\|_{L^2} \\ & \quad + \|\dot{\Delta}_j(B \cdot \nabla A)\|_{L^2} + \|R_j^3\|_{L^2} + \|R_j^4\|_{L^2}) \|B_j\|_{L^2}^2 \\ & \quad + \|\operatorname{div} E\|_{L^\infty} \|w_j\|_{L^2} \|B_j\|_{L^2}. \end{aligned} \tag{B.4}$$

By Propositions A.2 and A.3 and the commutator estimates in [1], we have the following estimates:

$$\begin{aligned} \|\dot{\Delta}_j(w \cdot \nabla A)\|_{L^2} &\leq C c_j 2^{-js} \|\nabla A\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|w\|_{\dot{B}_{2,1}^s}, \\ \|\dot{\Delta}_j(B \cdot \nabla E)\|_{L^2} &\leq C c_j 2^{-js} \|\nabla E\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|B\|_{\dot{B}_{2,1}^s}, \end{aligned}$$

$$\begin{aligned}
\|\dot{\Delta}_j(B \cdot \nabla A)\|_{L^2} &\leq C c_j 2^{-js} \|\nabla A\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|B\|_{\dot{B}_{2,1}^s}, \\
\|\dot{\Delta}_j(w \cdot \nabla E)\|_{L^2} &\leq C c_j 2^{-js} \|\nabla E\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|w\|_{\dot{B}_{2,1}^s}, \\
\|R_j^1\|_{L^2} &\leq C c_j 2^{-js} \|\nabla A\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|w\|_{\dot{B}_{2,1}^s}, \\
\|R_j^2\|_{L^2} &\leq C c_j 2^{-js} \|\nabla E\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|B\|_{\dot{B}_{2,1}^s}, \\
\|R_j^3\|_{L^2} &\leq C c_j 2^{-js} \|\nabla A\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|B\|_{\dot{B}_{2,1}^s}, \\
\|R_j^4\|_{L^2} &\leq C c_j 2^{-js} \|\nabla E\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|w\|_{\dot{B}_{2,1}^s},
\end{aligned}$$

where $(c_j)_{j \in \mathbb{Z}}$ denotes a positive sequence such that $\sum_{j \in \mathbb{Z}} c_j = 1$.

Formally dividing both sides of the inequality (B.4) by $\|w_j\|_{L^2} + \|B_j\|_{L^2}$ and integrating over $[0, t]$ yields

$$\begin{aligned}
&\|w_j(t)\|_{L^2} + \|B_j(t)\|_{L^2} + \mathcal{L} 2^{2j} \int_0^t (\|w_j(\tau)\|_{L^2} + \|B_j(\tau)\|_{L^2}) d\tau \\
&\leq \|w_j(0)\|_{L^2} + \|B_j(0)\|_{L^2} + \int_0^t (\|f_j(\tau)\|_{L^2} + \|g_j(\tau)\|_{L^2}) d\tau \\
&\quad + C \int_0^t (\|\nabla A\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla E\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) (\|w_j(\tau)\|_{L^2} + \|B_j(\tau)\|_{L^2}) d\tau \\
&\quad + 2^{-js} C \int_0^t (\|\nabla A\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla E\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) (\|w\|_{\dot{B}_{2,1}^s} + \|B\|_{\dot{B}_{2,1}^s}) d\tau. \tag{B.5}
\end{aligned}$$

Now, multiplying the both sides of (B.5) by 2^{js} and summing over j , we end up with

$$\begin{aligned}
&\|w\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^s)} + \|B\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^s)} + \kappa \mathcal{L} \|w\|_{L_t^1(\dot{B}_{2,1}^{s+2})} + \kappa \mathcal{L} \|B\|_{L_t^1(\dot{B}_{2,1}^{s+2})} \\
&\leq \|w_0\|_{\dot{B}_{2,1}^s} + \|B_0\|_{\dot{B}_{2,1}^s} + \|f\|_{L_t^1(\dot{B}_{2,1}^s)} + \|g\|_{L_t^1(\dot{B}_{2,1}^s)} \\
&\quad + C \int_0^t (\|\nabla A\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla E\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) (\|w\|_{\dot{B}_{2,1}^s} + \|B\|_{\dot{B}_{2,1}^s}) d\tau
\end{aligned}$$

for some constant C depending only on d and s . Applying Gronwall's lemma then completes the proof. \square

If (w, B) solve the following systems

$$\begin{cases} \partial_t w - \mu \Delta w + \mathcal{P}(A \cdot \nabla w) + \mathcal{P}(w \cdot \nabla A) \\ \quad - \mathcal{P}(B \cdot \nabla E) - \mathcal{P}(E \cdot \nabla B) = \mathcal{P}f, \\ \partial_t B - \nu \Delta B + \mathcal{P}(A \cdot \nabla B) - \mathcal{P}(B \cdot \nabla A) \\ \quad + \mathcal{P}(w \cdot \nabla E) - \mathcal{P}(E \cdot \nabla w) = \mathcal{P}g, \\ \operatorname{div} w = 0, \quad \operatorname{div} B = 0, \\ (w, B)|_{t=0} = (w_0, B_0)(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{B.6}$$

we have

Proposition B.3. *Let $s \in (-\frac{d}{2}, \frac{d}{2}]$. Let $s \in (-\frac{d}{2}, \frac{d}{2}]$. Assume that $w_0, B_0 \in \dot{B}_{2,1}^s$ with $\operatorname{div} w_0 = \operatorname{div} B = 0$, $f, g \in L_T^1(\dot{B}_{2,1}^s)$, and $A, E \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})$ are time-dependent vector fields. Then there exists a universal constant κ , and a constant C depending only on d and s , such that, for all $t \in [0, T]$,*

$$\begin{aligned} & \| (w, B) \|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^s)} + \kappa \underline{\nu} \| (w, B) \|_{L_t^1(\dot{B}_{2,1}^{s+2})} \\ & \leq (\| (w_0, B_0) \|_{\dot{B}_{2,1}^s} + \| (f, g) \|_{L_t^1(\dot{B}_{2,1}^s)}) \exp \left\{ C \int_0^t (\| \nabla A \|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \| \nabla E \|_{\dot{B}_{2,1}^{\frac{d}{2}}}) d\tau \right\} \end{aligned}$$

with $\underline{\nu} := \min\{\mu, \nu\}$.

Proof. The proof is similar to that of Proposition B.2. The evolution equations for $(w_j, B_j) := (\dot{\Delta}_j w, \dot{\Delta}_j B)$ now read

$$\begin{cases} \partial_t w_j - \mu \Delta w_j + \mathcal{P}(A \cdot \nabla w_j) - \mathcal{P}(E \cdot \nabla B_j) \\ \quad = \mathcal{P} f_j - \dot{\Delta}_j \mathcal{P}(w \cdot \nabla A) + \dot{\Delta}_j \mathcal{P}(B \cdot \nabla E) + \mathcal{P} R_j^1 - \mathcal{P} R_j^2 \\ \partial_t B_j - \nu \Delta B_j + \mathcal{P} A \cdot \nabla B_j - \mathcal{P} E \cdot \nabla w_j \\ \quad = \mathcal{P} g_j + \dot{\Delta}_j \mathcal{P}(B \cdot \nabla A) - \dot{\Delta}_j \mathcal{P}(w \cdot \nabla E) + \mathcal{P} R_j^3 - \mathcal{P} R_j^4 \end{cases}$$

Since $\operatorname{div} w_j = 0$ and $\operatorname{div} B_j = 0$, we can deduce that

$$\int h \cdot w_j dx = \int \mathcal{P} h \cdot w_j dx$$

for any $h \in L^2(\mathbb{R}^d)$. Taking the L^2 inner product for the equations in (B.7) with w_j and B_j respectively, the operator \mathcal{P} may be “omitted” in the computations so that by proceeding along the lines of the proof of Proposition B.2, we get the desired inequality. \square

Remark B.1. In the case of $d = 2$, the existence time T in Propositions B.1, B.2, and B.3 may take $+\infty$. Since we mainly study the local solution, we shall not discuss the details here.

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